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An Expansion of the Bessel-Maitland Function and its Associated Properties

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Abstract

Certain interesting progress inspired like as generalized Pochhammer symbols and extension of the hypergeometric function in special functions, we consider an extension of the Bessel-Maitland (BM) function $J_{d,\eta,\xi,\zeta,\alpha}^{c,a,b,f,\sigma}(y_1)$ express in terms of extended Pochhammer symbol. Consciously, we establish special properties, relations and checked the behavior of higher order differentials of the state function extension of generalized (BM-IV) function. Also, present the integral representation of main function $J_{d,\eta,\xi,\zeta,\alpha}^{c,a,b,f,\sigma}(y_1)$ and obtained the key results.

Keywords: Bessel-Maitland, hypergeometric function, Pochhammer symbol, Integral transform.

1 Introduction

In 1935 Wright, (1935)introduced the generalization of Bessel (BI) function by way of series representation said as Bessel-Maitland (BM-I) function defined as follows:

$$\mathrm{J}_d^c(y_1) = \sum_{t=0}^{\infty} \frac{(-y_1)^t}{\Gamma(ct+d+1)t!} \tag{1}$$

In 1995 Watson in his book "A treatise on the theory of Bessel functions"(Watson, 1922) . discussed the (BM-I) applications in the diverse area of biological, mathematical physics, engineering, and chemical in detail manners. Pathak, (1966) described the generalization of (BM-I) by adding nominator terms in Pochhammer symbol $(\eta)_{at}$ and

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specified as generalized Bessel-Maitland (BM-II) function for $c, d, \eta \in \mathbb{C}$, $\Re(c) \geq 0, \Re(d) \geq -1, \Re(\eta) \geq 0, a \in (0,1) \cup \mathbb{N}$ as follows:

$$\mathrm{J}_{d,a}^{c,\eta}(y_1) = \sum_{t=0}^{\infty} \frac{(\eta)_{at}(-y_1)^t}{\Gamma(ct+d+1)t!} \tag{2}$$

and properties of (BM-II) discussed by Singh et al. in (2014) and also gained results formation in Wright hypergeometric function (Zayed, 2021). In Ghayasuddin et al. (2018) nominated the extension of (BM-II) by adding one more denominator terms in Pochhammer symbol $(\xi)_{bt}$ instead of factorial form and presented the name as extended Bessel-Maitland (BM-III) function for $c, d, \eta, \xi \in \mathbb{C}, \Re(a) > 0, \Re(d) > -1$, $\Re(\eta) > 0, \Re(\xi) > 0$, and $a, b \in \mathbb{R}^+$with $a < \Re(c) + b$ as follows:

$$\mathrm{J}_{d,\eta,\xi}^{c,a,b}(y_1) = \sum_{t=0}^{\infty} \frac{(\eta)_{at}(-y_1)^t}{\Gamma(ct+d+1)(\xi)_{bt}} \tag{3}$$

The relations of (BM-III) with Mittag-Leffler (ML) function, differentials and integral representation of (BM-III) function pointed. In Ali et al. (2020) 66 developed the generalized Bessel-Maitland (BM-IV) function (eight-parameters) by inserting one more $(\zeta)_{ft}$ Pochhammer symbol in nominator (Usman et al., 2023). And presented the particular valid cases of (BM-IV) with (BM-1, BM-II, BM-III, ML) functions and considered the fractional integral operator involving (BM-IV) as kernel, which located in fractional calculus theory. The (BM-IV) function described as follows:

$$J_{d,\eta,\xi,\zeta}^{c,a,b,f}(y_1) = \sum_{t=0}^{\infty} \frac{(\eta)_{at}(\zeta)_{ft}(-y_1)^t}{\Gamma(ct+d+1)(\xi)_{bt}} \tag{4}$$

where $c, d, \eta, \xi, \zeta \in \mathbb{C}, \Re(a) > 0, \Re(d) > -1$ with $\Re(\eta) > 0, \Re(\xi) > 0, \Re(\zeta) > 0$, and $a, f, b \geq 0, f, a > \Re(c) + b$. Pochhammer symbols [7-10] represented as follows:

$$(\eta)_a = \begin{cases} 1, & a = 0, \eta \neq 0 \\ \eta(\eta+1)(\eta+2)\cdots(\eta+a-1), & a \geq 1 \end{cases} \tag{5}$$

In 2014 Srivastava et. al [11 defined the generalized Pochhammer symbol $(\eta; p)_a$ for $\eta, a \in \mathbb{C}$ and integral representation of $(\eta; p)_a$, where the $\Gamma_p(y_1)$ related to the third kind Modified Bessel function $[12-17]$. Pochhammer symbol $(\eta; p)_a$ formation as follows:

$$(\eta; p)_a = \begin{cases} \dfrac{\Gamma_p(\eta+a)}{\Gamma(a)}, & (\Re(p) > 0; \eta, a \in \mathbb{C}) \\ (\eta)_a, & p = 0; \eta, a \in \mathbb{C} \end{cases} \tag{6}$$

Pochhammer symbol $(\eta; p)_a$ applications and properties 18 extensions also mentioned. In 2019 Srivastava et al. (2019) defined extension of Pochhammer symbols $(\rho; \varrho, \sigma)_\alpha$ and explained integral representation of $(\rho; \varrho, \sigma)_\alpha$ written as follows respectively:

$$(\rho;\varrho,\sigma)_\alpha = \begin{cases} \frac{\Gamma_\sigma(\rho+\alpha;\varrho)}{\Gamma(\alpha)}, & (\Re(\varrho),\Re(\sigma)>0;\rho,\alpha\in\mathbb{C}) \\ (\rho;\varrho)_\alpha, & \sigma=0,\rho,\alpha\in\mathbb{C}/\{0\} \end{cases} \tag{7}$$

$$(\rho;\varrho,\sigma)_\alpha = \sqrt{\frac{2\varrho}{\pi}}\frac{1}{\Gamma(\rho)}\int_0^\infty y_1^{\ \rho+\alpha-\frac{3}{2}}\exp\left(-y_1\right)K_{\alpha+\frac{1}{2}}\left(\frac{\varrho}{y_1}\right)dy_1 \tag{8}$$

In integral $K_\alpha(.) is order \alpha$ modified Bessel function (Choi & Agarwal, 2013) . The same and variations

In $(\rho;\varrho,\sigma)_\alpha$ that the form of extension is also generalized and utilized at (Rahman et al., 2020;

Safdar et al., 2019;Sahai & Verma, 2016; Şahin & Yağcı, 2020).

Related of $(\rho;\varrho,\sigma)_\alpha$ generalized hypergeometric function 11, 25, Guess hypergeometric function ${}_2F_1$ and confluent hypergeometric function ${}_1F_1$ 26 respectively represented as follows:

$${}_rF_s = [(\rho_1;\varrho,\sigma)\cdots(\rho_r);(\alpha_1)\cdots(\alpha_s);y_1]$$

where $\rho_j\in\mathbb{C}$ for $j=1,2\cdots r,\alpha_i\in\mathbb{C}$ for $i=1,2\cdots s$ and $\alpha_i\neq 0,-1,-2\cdots$.

$${}_2F_1[(\rho_1;\varrho,\sigma),d;\alpha;y_1] = \sum_{t=0}^{\infty}\frac{(\rho_1;\varrho,\sigma)_t(d)_t}{(\alpha)_t}\frac{(y_1)^t}{t!} \tag{10}$$

and

$${}_1F_1[(\rho_1;\varrho,\sigma);\alpha;y_1] = \Phi[(\rho_1;\varrho,\sigma);\alpha;y_1] = \sum_{t=0}^{\infty}\frac{(\rho_1;\varrho,\sigma)_t}{(\alpha)_t}\frac{(y_1)^t}{t!} \tag{11}$$

Various applications and special properties of newly extensions of special functions in numerous field defined (Rahman et al., 2019; Mubeen et al., 2020; Chaudhry & Zubair, 2001). Inspired by the above mention extensions of Pochhammers symbols in special functions (Kulmitra & Tiwari, 2024), we present an extension of (BL-IV) function (4) in expression of the Pochhammer symbol (7) and examine its variations (Goyal et al., 2021) .

2 (BL-IV) Function Extension

Extension of the generalized (BL-IV) function (4) with Pochhammer symbol (6) and generalized Pochhammer symbol (6) defined in the form of definitions.

Definition 2.1. An extension of the generalized (BL-IV) function in (4) with Pochhammer symbol (6) represented as follows:

$$\mathrm{J}_{d,\eta,\xi,\xi}^{c,a,b,f,\sigma}(y_1)=\sum_{t=0}^{\infty}\frac{(\eta;\sigma)_{at}(\zeta;\sigma)_{ft}(-y_1)^t}{\Gamma(ct+d+1)(\xi;\sigma)_{bt}t!} \tag{12}$$

where $c,d,\eta,\xi,\zeta\in\mathbb{C},\Re(\sigma)>0,\Re(a)>0,\Re(d)>-1$ with $\Re(\eta)>0,\Re(\xi)>0$, $\Re(\zeta)>0$, and $a,f,b,\sigma\geq 0,f,a>\Re(c)+b$ given series is right sided converges. If we insert $\sigma=0$ in (12) then it reduce to generalized (BL-IV) of (4).

Definition 2.2. An extension of the generalized (BL-IV) function in (4) with generalized Pochhammer symbol (7) represented as follows:

$$\mathrm{J}_{d,\eta,\xi,\zeta,\alpha}^{c,a,b,f,\sigma}(y_1)=\sum_{t=0}^{\infty}\frac{(\eta;\sigma,\alpha)_{at}(\zeta;\sigma,\alpha)_{ft}(-y_1)^t}{\Gamma(ct+d+1)(\xi;\sigma,\alpha)_{bt}t!} \tag{13}$$

where $c,d,\eta,\xi,\zeta\in\mathbb{C},\Re(\sigma)>0,\Re(a)>0,\Re(d)>-1$ with $\Re(\eta)>0,\Re(\xi)>0$, $\Re(\zeta)>0$, and $a,f,b,\sigma,\alpha\geq 0,f,a>\Re(c)+b$ given series is right sided converges.

If we insert $\alpha=0$ in (13) then it reduce to defined (12). And inserting $a=1,f=b=0,d=d-1$ and y_1 replace by $-y_1$ then it reduce to Mittag-Leffler function defined in (Rahman et al., 2020). We put $a=1,f=b=0,d=d-1$ and y_1 replace by $-y_1$, $\alpha=0$ then it reduce to defined function in (Choi et al., 2020).

If we insert $a=c=1,f=b=0$ and replacing y_1 by $-y_1$ then (13) reduced to confluent hypergeometric function as:

$$J_{d,\eta,\xi,\zeta,\alpha}^{1,1,0,0,\sigma}(-y_1)=\frac{1}{\Gamma(d+1)}\Phi[(\eta;\sigma,\alpha);d+1;y_1]$$

3 $J_{d,\eta,\xi,\zeta,\alpha,\alpha}^{c,a,b,\sigma}(y_1)$ Function Properties

In this section, we specified the certain particular properties and integral representations of the (BL-IV) extended function $J_{d,\eta,\xi,\zeta,\alpha}^{c,a,b,f,\sigma}(y_1)$ in 13.

Theorem 3.1. The following relations holds for the function $J_{d,\eta,\xi,\xi,\zeta,\alpha}^{c,a,b,f}(y_1)$ in 13 . for $c,d,\eta,\xi,\zeta\in\mathbb{C},\Re(\sigma)>0,\Re(a)>0,\Re(d)>-1$ with $\Re(\eta)>0,\Re(\xi)>0$, $\Re(\zeta)>0$, and $a,f,b,\sigma,\alpha\geq 0,f,a>\Re(c)+b$ as

$$J_{d,\eta,\xi,\zeta,\alpha}^{c,a,b,f,\sigma}(y_1)=(d+1)J_{d+1,\eta,\xi,\zeta,\alpha}^{c,a,b,f,\sigma}(y_1)+c(y_1)\frac{d}{dy_1}J_{d+1,\eta,\xi,\zeta,\alpha}^{c,a,b,f,\sigma}(y_1) \tag{15}$$

Accurately, we attain

$$J_{d,\eta,\xi,\zeta}^{c,a,b,f,\sigma}(y_1)=(d+1)J_{d+1,\eta,\xi,\zeta}^{c,a,b,f,\sigma}(y_1)+c(y_1)\frac{d}{dy_1}J_{d+1,\eta,\xi,\zeta}^{c,a,b,f,\sigma}(y_1) \tag{16}$$

Proof. Consider from the equation (13), we get

$$(d+1)J_{d+1,\eta,\xi,\xi,\alpha}^{c,a,b,f,\alpha}(y_1)+c(y_1)\frac{d}{dy_1}J_{d+1,\eta,\xi,\zeta,\alpha}^{c,a,b,f,}(y_1)$$
$$=(d+1)\sum_{t=0}^{\infty}\frac{(\eta;\sigma,\alpha)_{at}(\zeta;\sigma,\alpha)_{ft}(-y_1)^t}{\Gamma(ct+d+2)(\xi;\sigma,\alpha)_{bt}t!}+c(y_1)\frac{d}{dy_1}\sum_{t=0}^{\infty}\frac{(\eta;\sigma,\alpha)_{at}(\zeta;\sigma,\alpha)_{ft}(-y_1)^t}{\Gamma(ct+d+2)(\xi;\sigma,\alpha)_{bt}t!}$$
$$=(d+1)\sum_{t=0}^{\infty}\frac{(\eta;\sigma,\alpha)_{at}(\zeta;\sigma,\alpha)_{ft}(-y_1)^t}{\Gamma(ct+d+2)(\xi;\sigma,\alpha)_{bt}t!}+c(y_1)\sum_{t=0}^{\infty}\frac{(\eta;\sigma,\alpha)_{at}(\zeta;\sigma,\alpha)_{ft}(-1)^t[t(y_1)^{t-1}]}{\Gamma(ct+d+2)(\xi;\sigma,\alpha)_{bt}t!}$$
$$=(d+1)\sum_{t=0}^{\infty}\frac{(\eta;\sigma,\alpha)_{at}(\zeta;\sigma,\alpha)_{ft}(-y_1)^t}{\Gamma(ct+d+2)(\xi;\sigma,\alpha)_{bt}t!}+\sum_{t=0}^{\infty}\frac{ct(\eta;\sigma,\alpha)_{at}(\zeta;\sigma,\alpha)_{ft}(-y_1)^t}{\Gamma(ct+d+2)(\xi;\sigma,\alpha)_{bt}t!}$$
$$=\sum_{t=0}^{\infty}\frac{(\eta;\sigma,\alpha)_{at}(\zeta;\sigma,\alpha)_{ft}(ct+d+1)(-y_1)^t}{(ct+d+1)\Gamma(ct+d+1)(\xi;\sigma,\alpha)_{bt}t!} using \Gamma(y_1+1)=y_1\Gamma(y_1)$$
$$=\sum_{t=0}^{\infty}\frac{(\eta;\sigma,\alpha)_{at}(\zeta;\sigma,\alpha)_{ft}(-y_1)^t}{\Gamma(ct+d+1)(\xi;\sigma,\alpha)_{bt}t!}=J_{d,\eta,\xi,\zeta,\alpha}^{c,a,b,f,\sigma}(y_1)$$

If we insert $\alpha=0$ in equation (15) then we get the desired result of equation (16) (Sachan et al., 2024) .

Theorem 3.2. The following formula of higher differential holds for the function $J_{d,\eta,\xi,\zeta,\alpha}^{c,a,b,f,\sigma}(y_1)$ in 13) for $c,d,\eta,\xi,\zeta,\lambda\in\mathbb{C},\Re(\sigma)>0,\Re(a)>0,\Re(d)>-1$ with $\Re(\eta)>0,\Re(\xi)>0,\Re(\zeta)>0$, and $a,f,b,\sigma,\alpha\geq 0,f,a>\Re(c)+b$ as

$$\left(\frac{d}{dy_1}\right)^m \mathrm{J}_{d,\eta,\xi,\zeta,\alpha}^{c,a,b,f,\sigma}(y_1)=\frac{(\eta)_{am}(\zeta)_{fm}}{(\xi)_{bm}(-1)^{-m}}J_{d+cm,\eta+am,\xi+bm,\zeta+fm,\alpha}^{c,a,b,f,\sigma}(y_1) \tag{17}$$
$$\left(\frac{d}{dy_1}\right)^m\left[(y_1)^d\,\mathrm{J}_{d,\eta,\xi,\zeta,\alpha}^{c,a,b,f,\sigma}(\lambda y_1^c)\right]=(y_1)^{d-m}\,\mathrm{J}_{d-m,\eta,\xi,\zeta,\alpha}^{c,a,b,f,\sigma}(\lambda y_1^c) \tag{18}$$

for $\Re(d-m)>0,m\in\mathbb{N}$.

$$\left(\frac{d}{dy_1}\right)^m[(y_1)^d\Phi((\eta;\sigma,\alpha);d+1;\lambda y_1)]$$

Accurately, we get

$$\left(\frac{d}{dy_1}\right)^m J_{d,\eta,\xi,\zeta}^{c,a,b,f,\sigma}(y_1)=\frac{(\eta)_{am}(\zeta)_{fm}}{(\xi)_{bm}(-1)^{-m}}J_{d+cm,\eta+am,\xi+bm,\zeta+fm}^{c,a,b,f,\sigma}(y_1) \tag{20}$$

$$\left(\frac{d}{dy_1}\right)^m \left[(y_1)^d \, \mathrm{J}_{d,\eta,\xi,\xi}^{c,a,b,f,\sigma}(\lambda y_1^c)\right] = (y_1)^{d-m} \, \mathrm{J}_{d-m,\eta,\xi,\zeta}^{c,a,b,f,\sigma}(\lambda y_1^c) \tag{21}$$

$$\left(\frac{d}{dy_1}\right)^m [(y_1)^d \Phi((\eta;\sigma); d+1; \lambda y_1)] \tag{21}$$

$$= \frac{\Gamma(d+1)}{\Gamma(d-m+1)}(y_1)^{d-m}\Phi((\eta;\sigma); d-m+1; \lambda y_1) \tag{22}$$

Proof. Terms wise operating differentiations of (13) m times, we attain

$$\left(\frac{d}{dy_1}\right)^m J_{d,\eta,\xi,\zeta,\alpha}^{c,a,b,f,\sigma}(y_1) = \left(\frac{d}{dy_1}\right)^m \sum_{t=0}^{\infty} \frac{(\eta;\sigma,\alpha)_{at}(\zeta;\sigma,\alpha)_{ft}(-1)^t (y_1)^t}{\Gamma(ct+d+1)(\xi;\sigma,\alpha)_{bt} t!}$$

Replacing t by $t+m$

$$\left(\frac{d}{dy_1}\right)^m J_{d,\eta,\xi,\zeta,\alpha}^{c,a,b,f,\sigma}(y_1) = \sum_{t=0}^{\infty} \frac{(\eta;\sigma,\alpha)_{a(t+m)}(\zeta;\sigma,\alpha)_{f(t+m)}(-1)^{t+m}(y_1)^t}{\Gamma(ct+cm+d+1)(\xi;\sigma,\alpha)_{b(t+m)}(t)!} \tag{24}$$

Using $(\eta;\sigma,\alpha)_{am+at} = (\eta)_{am}(\eta+am;\sigma,\alpha)_{at}$, then

$$\left(\frac{d}{dy_1}\right)^m J_{d,\eta,\xi,\zeta,\alpha}^{c,a,b,f,\sigma}(y_1) = \sum_{t=0}^{\infty} \frac{(\eta)_{am}(\eta+am;\sigma,\alpha)_{at}(\zeta)_{fm}(\zeta+fm;\sigma,\alpha)_{ft}(-1)^m(-y_1)^t}{\Gamma(ct+cm+d+1)(\xi)_{bm}(\xi+bm;\sigma,\alpha)_{bt}(t)!}$$
$$= \frac{(\eta)_{am}(\zeta)_{fm}}{(\xi)_{bm}(-1)^{-m}} \sum_{t=0}^{\infty} \frac{(\eta+am;\sigma,\alpha)_{at}(\zeta+fm;\sigma,\alpha)_{ft}(-y_1)^t}{\Gamma(ct+cm+d+1)(\xi+bm;\sigma,\alpha)_{bt}(t)!}$$

Similar manners for $\sqrt{18}$, we have

$$\left(\frac{d}{dy_1}\right)^m \left[(y_1)^d \, \mathrm{J}_{d,\eta,\xi,\zeta,\alpha}^{c,a,b,f,\sigma}(\lambda y_1^c)\right] = \left(\frac{d}{dy_1}\right)^m \left[(y_1)^d \sum_{t=0}^{\infty} \frac{(\eta;\sigma,\alpha)_{at}(\zeta;\sigma,\alpha)_{ft}(-\lambda y_1^c)^t}{\Gamma(ct+d+1)(\xi;\sigma,\alpha)_{bt} t!}\right]$$
$$= \left(\frac{d}{dy_1}\right)^m \left[\sum_{t=0}^{\infty} \frac{(\eta;\sigma,\alpha)_{at}(\zeta;\sigma,\alpha)_{ft}(-\lambda)^t (y_1)^{ct+d}}{\Gamma(ct+d+1)(\xi;\sigma,\alpha)_{bt} t!}\right]$$
$$= \sum_{t=0}^{\infty} \frac{(\eta;\sigma,\alpha)_{at}(\zeta;\sigma,\alpha)_{ft}(-\lambda)^t (ct+d)!\,(y_1)^{ct+d-m}}{\Gamma(ct+d+1)(\xi;\sigma,\alpha)_{bt}(ct+d-m)!\,t!}$$
$$= \sum_{t=0}^{\infty} \frac{(\eta;\sigma,\alpha)_{at}(\zeta;\sigma,\alpha)_{ft}(-\lambda)^t \Gamma(ct+d+1)(y_1)^{ct+d-m}}{\Gamma(ct+d+1)(\xi;\sigma,\alpha)_{bt}\Gamma(ct+d-m+1)t!}$$
$$= (y_1)^{d-m} \sum_{t=0}^{\infty} \frac{(\eta;\sigma,\alpha)_{at}(\zeta;\sigma,\alpha)_{ft}(-\lambda y_1^c)^t}{(\xi;\sigma,\alpha)_{bt}\Gamma(ct+d-m+1)t!}$$
$$= (y_1)^{d-m} \, \mathrm{J}_{d-m,\eta,\xi,\zeta,\alpha}^{c,a,b,f,\sigma}(\lambda y_1^c)$$

Moreover, inserting $a = c = 1, f = b = 0$ in resulting equation (18) gives equation result of (19) (Ghanim et al., 2024) . Special cases of (17), 18), and (19), when we insert $\alpha = 0$, we get (20), 21), and 22 , respectively.

Corollary 3.1. Integral representation of (BM) function $J_{d,\eta,\xi,\zeta,\alpha}^{c,a,b,f,\sigma}(y_1)$ 13) holds in the following manners:

$$\int_0^{y_1} l^d \, \mathrm{J}_{d,\eta,\xi,\zeta,\alpha}^{c,a,b,f,\sigma}(\lambda l^c) dl = (y_1)^{d+1} \, \mathrm{J}_{d+1,\eta,\xi,\zeta,\alpha}^{c,a,b,f,\sigma}(\lambda y_1^c) \tag{26}$$

$$\int_0^{y_1} l^d \frac{1}{\Gamma(d+1)} \Phi((\eta;\sigma,\alpha); d+1; \lambda l) dl \tag{26}$$

$$= \frac{1}{\Gamma(d+2)} (y_1)^{d+1} \Phi((\eta;\sigma,\alpha); d+2; \lambda y_1) \tag{27}$$

where $c, d, \eta, \xi, \zeta, \lambda \in \mathbb{C}, \Re(\sigma) > 0, \Re(a) > 0, \Re(d) > -1$ with $\Re(\eta) > 0, \Re(\xi) > 0$, $\Re(\zeta) > 0$, and $a, f, b, \sigma, \alpha \geq 0, f, a > \Re(c) + b$.

Accurately, we get

$$\int_0^{y_1} l^d \, \mathrm{J}_{d,\eta,\xi,\zeta}^{c,a,b,f,\sigma}(\lambda l^c) dl = (y_1)^{d+1} \, \mathrm{J}_{d+1,\eta,\xi,\zeta}^{c,a,b,f,\sigma}(\lambda y_1^c) \tag{28}$$

$$\int_0^{y_1} l^d \frac{1}{\Gamma(d+1)} \Phi((\eta;\sigma); d+1; \lambda l) dl \tag{28}$$

$$= \frac{1}{\Gamma(d+2)} (y_1)^{d+1} \Phi((\eta;\sigma); d+2; \lambda y_1) \tag{29}$$

4 $J_{d,\eta,\xi,\zeta,\alpha}^{c,a,b,f,\sigma}(y_1)$ Function Representation in the Form of Generalized Hypergeometric Function

Here, we present the representation of $J_{d,\eta,\xi,\zeta,\alpha}^{c,a,b,f,\sigma}(y_1)$ function $\sqrt{13}$ in the form of Hypergeometric (Abro et al., 2021) function in generalized terms as follows:

Theorem 4.1. Defined function $J_{d,\eta,\xi,\zeta,\alpha}^{c,a,b,f,\sigma}(y_1)$ in 13 for $c \in \mathbb{N}$ represented as in the form of generalized Hypergeometric function as follows:

$$\mathrm{J}_{d,\eta,\xi,\xi,\alpha}^{r,a,b,f,\sigma}(y_1) = \frac{1}{\Gamma(d+1)} F_r \left[(\eta;\sigma,\alpha)_a, (\zeta;\sigma,\alpha)_f; \frac{d+1}{r}, \frac{d+2}{r} \cdots \frac{d+r}{r}, (\xi;\sigma,\alpha)_b; \frac{y_1}{r} \right]$$

where the parameters $\frac{d+1}{r}, \frac{d+2}{r} \cdots \frac{d+r}{r}$ in $\Delta(r; d+1)$ is is an array of ${}_pf_r$ and $r \in \mathbb{N}$.

Proof. Inserting $c = r \in \mathbb{N}$ in (13) and utilizing the gamma function formula of multiplication, we get

$$J^{r,a,b,f,\sigma}_{d,\eta,\xi,\zeta,\alpha}(y_1) = \sum_{t=0}^{\infty} \frac{(\eta;\sigma,\alpha)_{at}(\zeta;\sigma,\alpha)_{ft}(-y_1)^t}{\Gamma(rt+d+1)(\xi;\sigma,\alpha)_{bt}t!}$$
$$= \frac{1}{\Gamma(d+1)}\sum_{t=0}^{\infty} \frac{(\eta;\sigma,\alpha)_{at}(\zeta;\sigma,\alpha)_{ft}(-y_1)^t}{\Gamma(rt+d+1)/\Gamma(d+1)(\xi;\sigma,\alpha)_{bt}t!}$$

Since $(d+1)_{rt} = \prod_{j=1}^{r}\left(\frac{d+j}{r}\right)_t \times r^{rt}$

$$J^{r,a,b,f,\sigma}_{d,\eta,\xi,\zeta,\alpha}(y_1) = \frac{1}{\Gamma(d+1)}\sum_{t=0}^{\infty} \frac{(\eta;\sigma,\alpha)_{at}(\zeta;\sigma,\alpha)_{ft}\left(\frac{-y_1}{r}\right)^t}{\prod_{j=1}^{r}\left(\frac{d+j}{r}\right)_t (\xi;\sigma,\alpha)_{bt}t!}$$
$$= \frac{1}{\Gamma(d+1)}\sum_{t=0}^{\infty} \frac{(\eta;\sigma,\alpha)_{at}(\zeta;\sigma,\alpha)_{ft}}{\left[\left(\frac{d+1}{r}\right)_t,\left(\frac{d+2}{r}\right)_t \cdots \left(\frac{d+r}{r}\right)_t\right](\xi;\sigma,\alpha)_{bt}}\left(\frac{-y_1}{r}\right)^t \frac{1}{t!}$$
$$= \frac{1}{\Gamma(d+1)} F_r\left[(\eta;\sigma,\alpha)_a,(\zeta;\sigma,\alpha)_f;\frac{d+1}{r},\frac{d+2}{r}\cdots\frac{d+r}{r},(\xi;\sigma,\alpha)_b;\frac{y_1}{r}\right]$$

5 $J^{c,a,b,f,\sigma}_{d,\eta,\xi,\zeta,\alpha}(y_1)$ Function Integral Representations

Here, we establish numerous integral representation of the $J^{c,a,b,f,\sigma}_{d,\eta,\xi,\zeta,\alpha}(y_1)$ function in 13 such as the Euler transform, the Mellin transform, and the Laplace transform.

Definition: Euler-beta transform for the function $f(y_1)$ defined in [6] as follows:

$$\mathrm{B}\{f(y_1);c_1,d_1\} = \int_0^1 (y_1)^{c_1-1}(1-y_1)^{d_1-1}f(y_1)dy_1 \tag{33}$$

Theorem 5.1. Euler-beta transform for the function $J^{c,a,b,f,\sigma}_{d,\eta,\xi,\zeta,\alpha}(y_1)$ in 13 holds:

$$\mathrm{B}\left\{\mathrm{J}^{c,a,f,\sigma}_{d,\eta,\xi,\xi,\zeta,\alpha}\left(zy_1^{\rho}\right);c_1,d_1\right\} = \frac{\Gamma(d_1)\Gamma(c_1)}{\Gamma(c_1+d_1)\Gamma(d+1)}$$

Proof. Using (13) in equation (33), we get

$$\mathrm{B}\left\{J_{d,\eta,\xi,\xi,\zeta,\alpha}^{c,a,b,\sigma}(zy_1{}^{\rho});c_1,d_1\right\}$$
$$=\int_0^1 (y_1)^{c_1-1}(1-y_1)^{d_1-1}\, J_{d,\eta,\xi,\xi,\zeta,\alpha}^{c,a,b,\sigma,\sigma}(zy_1{}^{\rho})dy_1$$
$$=\int_0^1 (y_1)^{c_1-1}(1-y_1)^{d_1-1}\sum_{t=0}^{\infty}\frac{(\eta;\sigma,\alpha)_{at}(\zeta;\sigma,\alpha)_{ft}(-zy_1{}^{\rho})^t}{\Gamma(ct+d+1)(\xi;\sigma,\alpha)_{bt}t!}dy_1$$
$$=\sum_{t=0}^{\infty}\frac{(\eta;\sigma,\alpha)_{at}(\zeta;\sigma,\alpha)_{ft}(-z)^t}{\Gamma(ct+d+1)(\xi;\sigma,\alpha)_{bt}t!}\int_0^1 (y_1)^{c_1+\rho t-1}(1-y_1)^{d_1-1}dy_1$$
$$=\sum_{t=0}^{\infty}\frac{(\eta;\sigma,\alpha)_{at}(\zeta;\sigma,\alpha)_{ft}(-z)^t}{\Gamma(ct+d+1)(\xi;\sigma,\alpha)_{bt}t!}\frac{\Gamma(c_1+\rho t)\Gamma(d_1)}{\Gamma(c_1+\rho t+d_1)}$$
$$=\frac{\Gamma(d_1)\Gamma(c_1)}{\Gamma(c_1+d_1)}\sum_{t=0}^{\infty}\frac{(\eta;\sigma,\alpha)_{at}(\zeta;\sigma,\alpha)_{ft}(-z)^t}{\Gamma(ct+d+1)(\xi;\sigma,\alpha)_{bt}t!}\frac{(c_1)_{\rho t}}{(d_1)_{\rho t}}$$

$$=\frac{\Gamma(d_1)\Gamma(c_1)}{\Gamma(c_1+d_1)\Gamma(d+1)}\sum_{t=0}^{\infty}\frac{(\eta;\sigma,\alpha)_{at}(\zeta;\sigma,\alpha)_{ft}(-z)^t}{(d+1)_{ct}(\xi;\sigma,\alpha)_{bt}t!}\frac{(c_1)_{\rho t}}{(d_1)_{\rho t}}$$
$$=\sum_{t=0}^{\infty}\frac{\Gamma(d_1)\Gamma(c_1)(\eta;\sigma,\alpha)_{at}(\zeta;\sigma,\alpha)_{ft}(-z)^t\prod_{j=1}^{\rho}\left((c_1+j-1)/\rho\right)_t\rho^{\rho t}}{\Gamma(d+1)\prod_{j=1}^{c}((d+j)/c)_t c^{ct}(\xi;\sigma,\alpha)_{bt}t!\,\Gamma(c_1+d_1)\prod_{j=1}^{\rho}(d_1+j-1/\rho)_t\rho^{\rho t}}$$
$$=\sum_{t=0}^{\infty}\frac{\Gamma(d_1)\Gamma(c_1)(\eta;\sigma,\alpha)_{at}(\zeta;\sigma,\alpha)_{ft}(-z)^t\prod_{j=1}^{\rho}\left((c_1+j-1)/\rho\right)_t}{\Gamma(d+1)\prod_{j=1}^{c}((d+j)/c)_t c^{ct}(\xi;\sigma,\alpha)_{bt}t!\,\Gamma(c_1+d_1)\prod_{j=1}^{\rho}(d_1+j-1/\rho)_t}$$
$$=\frac{\Gamma(d_1)\Gamma(c_1)}{\Gamma(c_1+d_1)\Gamma(d+1)}$$

Corollary 5.1. Inserting $c_1 = d$ and $\rho = c \in \mathbb{C}$ in (33) and then applying (13), we get

$$\int_0^1 (y_1)^{d}(1-y_1)^{d_1-1}\, J_{d,\eta,\xi,\zeta,\alpha}^{c,a,b,f,\sigma}(zy_1^c)dy_1=\Gamma(d_1)J_{d+d_1,\eta,\xi,\zeta,\alpha}^{c,a,b,f,\sigma}(z) \qquad (36)$$

Similarly, we attain

$$\int_0^1 (y_1)^{c_1-1}(1-y_1)^{d}\, J_{d,\eta,\xi,\zeta,\alpha}^{c,a,b,f,\sigma}(zy_1^c)dy_1=\Gamma(c_1)J_{d+c_1,\eta,\xi,\zeta,\alpha}^{c,a,b,f,\sigma}(z) \qquad (37)$$

generally, we attain

$$\int_0^1 (z-y_1)^{c_1-1}(y_1-x)^{d}J_{d,\eta,\xi,\zeta,\alpha}^{c,a,b,f,\sigma}(\lambda(y_1-x)^c)dy_1$$

5.2. Mellin transform. Integrable function $f(y_1)$ defined for index s as Mellin transform [26] represented as follows:

$$\mathrm{M}\{f(y_1): y_1 \to s\} = \int_0^\infty (y_1)^{s-1} f(y_1) dy_1 \tag{39}$$

(39) exist for improper integral.

Theorem 5.2. Mellin transform exist for the defined $\mathrm{J}_{d,\eta,\xi,\zeta,\alpha}^{c,a,b,f,\sigma}(y_1)$ function in 13 , as follows:

$$\mathrm{M}\left\{\mathrm{J}_{d,\eta,\xi,\zeta,\alpha}^{c,a,b,f,\sigma}(y_1): \alpha \to s\right\}$$

where $\Re(s-\alpha) > 0$, with $\Re(\eta+s) > -1, \Re(\xi+s) > -1, \Re(\zeta+s) > -1$.

Proof. Using the (13) in (39), we have a form

$$\mathrm{M}\left\{J_{d,\eta,\xi,\zeta,\alpha}^{c,a,b,f,\sigma}(y_1): \alpha \to s\right\}$$

$$\begin{aligned}
&= \int_0^\infty (\alpha)^{s-1}\, \mathrm{J}_{d,\eta,\xi,\xi,\zeta,\alpha}^{c,a,b,f}(y_1) d\alpha \\
&= \int_0^\infty (\alpha)^{s-1} \left(\sum_{t=0}^{\infty} \frac{(\eta;\sigma,\alpha)_{at}(\zeta;\sigma,\alpha)_{ft}(-y_1)^t}{\Gamma(ct+d+1)(\xi;\sigma,\alpha)_{bt} t!} \right) d\alpha \\
&= \int_0^\infty (\alpha)^{s-1} \left(\sum_{t=0}^{\infty} \frac{\Gamma(\xi)\Gamma_\alpha(\eta+at;\sigma,\alpha)\Gamma_\alpha(\zeta+ft;\sigma,\alpha)(-y_1)^t}{\Gamma(ct+d+1)\Gamma_\alpha(\xi+bt;\sigma,\alpha)\Gamma(\eta)\Gamma(\zeta)t!} \right) d\alpha
\end{aligned}$$

the order of integration and summation interchanging, we get

$$\mathrm{M}\left\{\mathrm{J}_{d,\eta,\xi,\zeta,\alpha}^{c,a,b,f,\sigma}(y_1): \alpha \to s\right\}$$

the result of [35] used, then

$$\begin{aligned}
&\mathrm{M}\left\{J_{d,\eta,\xi,\zeta,\alpha}^{c,a,b,f,\sigma}(y_1): \alpha \to s\right\} \\
&= \frac{2^{s-1}}{\sqrt{\pi}} \Gamma\left(\frac{s-\alpha}{2}\right) \Gamma\left(\frac{s+\alpha+1}{2}\right) \sum_{t=0}^{\infty} \frac{\Gamma(\xi)}{\Gamma(\eta)\Gamma(\zeta)} \frac{(-y_1)^t \Gamma(\eta+at+s)\Gamma(\zeta+ft+s)}{\Gamma(ct+d+1)t\Gamma\Gamma(\xi+bt+s)} \\
&= \frac{2^{s-1}}{\sqrt{\pi}} \Gamma\left(\frac{s-\alpha}{2}\right) \Gamma\left(\frac{s+\alpha+1}{2}\right) \sum_{t=0}^{\infty} \frac{(\eta)_s(\zeta)_s}{(\xi)_s} \frac{(-y_1)^t (\eta+s)_{at}(\zeta+s)_{ft}}{\Gamma(ct+d+1)t!\,(\xi+s)_{bt}} \\
&= \frac{2^{s-1}}{\sqrt{\pi}} \Gamma\left(\frac{s-\alpha}{2}\right) \Gamma\left(\frac{s+\alpha+1}{2}\right) \frac{(\eta)_s(\zeta)_s}{(\xi)_s} \sum_{t=0}^{\infty} \frac{(-y_1)^t (\eta+s)_{at}(\zeta+s)_{ft}}{\Gamma(ct+d+1)t!\,(\xi+s)_{bt}}
\end{aligned}$$

Corollary 5.2. The integral transform mention below exist:

$$\mathrm{M}\left\{\mathrm{J}_{d,\eta,\xi,\xi,\zeta,\alpha}^{c,a,b,f,\sigma}(y_1): \alpha \to s\right\}$$

where ${}_2\Psi_2$ is [6] Wright Hypergeometric function.

5.3. Laplace transform. Laplace transform for the function $f(y_1)$ defined in [6] as follows:

$$\mathrm{L}\{f(y_1)\} = \int_0^\infty \exp(-sy_1) f(y_1) dy_1 \tag{44}$$

Theorem 5.3. The Laplace transform holds for the function $J_{d,\eta,\xi,\zeta,\alpha}^{c,a,b,f,\sigma}(y_1)$ in 13 , as follows:

$$\mathbf{L}\left\{y_1^{c_1}\ \mathrm{J}_{d,\eta,\xi,\zeta,\alpha,\alpha}^{c,a,b,f}\left(\lambda y_1^{\rho}\right)\right\}$$

$$= \frac{\Gamma(c_1+1)}{s^{c_1+1}\Gamma(d+1)}\ {}^{\rho+1}F_c\left[(\eta;\sigma,\alpha)_a(\zeta;\sigma,\alpha)_f;\Delta(\rho;c_1+1);\Delta(c;d+1),(\xi;\sigma,\alpha)_b;\frac{-\lambda c^c}{(s\rho)^\rho}\right] \tag{45}$$

Proof. Using (13) in (44), we have

$$\begin{aligned}
&\mathrm{L}\left\{y_1^{c_1}\ \mathrm{J}_{d,\eta,\xi,\xi,\zeta,\alpha}^{c,a,\sigma,\sigma}\left(\lambda y_1^{\rho}\right)\right\} \\
&= \int_0^\infty \exp(-sy_1) y_1^{c_1} \mathbf{J}_{d,\eta,\xi,\xi,\zeta,\alpha}^{c,a,f,\sigma}\left(\lambda y_1^{\rho}\right) dy_1 \\
&= \int_0^\infty \exp(-sy_1) y_1^{c_1} \sum_{t=0}^{\infty} \frac{(\eta;\sigma,\alpha)_{at}(\zeta;\sigma,\alpha)_{ft}\left(-\lambda y_1^{\rho}\right)^t}{\Gamma(ct+d+1)(\xi;\sigma,\alpha)_{bt}t!} \\
&= \sum_{t=0}^{\infty} \frac{(\eta;\sigma,\alpha)_{at}(\zeta;\sigma,\alpha)_{ft}(-\lambda)^t}{\Gamma(ct+d+1)(\xi;\sigma,\alpha)_{bt}t!} \int_0^\infty \exp(-sy_1) y_1^{c_1+\rho t} dy_1 \\
&= \sum_{t=0}^{\infty} \frac{(\eta;\sigma,\alpha)_{at}(\zeta;\sigma,\alpha)_{ft}(-\lambda)^t}{\Gamma(ct+d+1)(\xi;\sigma,\alpha)_{bt}t!} \frac{\Gamma(c_1+\rho t+1)}{s^{c_1+\rho t+1}} \\
&= \frac{\Gamma(c_1+1)}{\Gamma(d+1)} \sum_{t=0}^{\infty} \frac{(\eta;\sigma,\alpha)_{at}(\zeta;\sigma,\alpha)_{ft}(-\lambda)^t}{(d+1)_{ct}(\xi;\sigma,\alpha)_{bt}t!} \frac{(c_1+1)_{\rho t}}{s^{c_1+\rho t+1}} \\
&= \frac{\Gamma(c_1+1)}{s^{c_1+1}\Gamma(d+1)} \sum_{t=0}^{\infty} \frac{(\eta;\sigma,\alpha)_{at}(\zeta;\sigma,\alpha)_{ft}(-\lambda c^c)^t}{(d+1)_{ct}(\xi;\sigma,\alpha)_{bt}t!} \frac{(c_1+1)_{\rho t}}{(s^{\rho}\rho^{\rho})^t}
\end{aligned}$$

Corollary 5.3. Inserting $c_1 = d$ and $\rho = c$ in (45), we have

$$\int_0^\infty \exp(-sy_1) y_1^{d} \mathrm{J}_{d,\eta,\xi,\xi,\zeta}^{c,a,b,f,\sigma}(\lambda y_1^{c}) dy_1$$

5.4. Whittaker transform. Use the following formation formula to find out the Whittaker transform of $J_{d,\eta,\xi,\zeta,\alpha}^{c,a,b,f,\sigma}(y_1)$ as follows:

$$\int_0^\infty y^{\alpha-1}\exp\left(-(1/2)t\right)\mathrm{W}_{\xi,\eta}dy=\frac{\Gamma((2)\pm\eta+\alpha)}{\Gamma(1-\xi+\alpha)}\left(\Re\left(\alpha\pm\eta>-\frac{1}{2}\right)\right) \tag{48}$$

Theorem 5.4. The Whittaker transform holds for the function $\int_{d,\eta,\xi,\zeta,\alpha}^{c,a,b,f,\sigma}(y_1)$ in 13), as follows:

$$\int_0^\infty y_1^{\beta-1}\exp\left(-(1/2)sy_1\right)\mathrm{W}_{\xi,\eta}(sy_1)\mathrm{J}_{d,\eta,\xi,\zeta,\alpha}^{c,a,b,f,\sigma}\left(\lambda y_1^{\vartheta}\right)$$
$$=\frac{\Gamma((2)\pm\eta+\beta)}{\Gamma(1-\xi+\beta)\Gamma(d+1)s^{\beta}}$$

where $c,d,\eta,\xi,\zeta,\vartheta,\beta\in\mathbb{C},\Re(\sigma)>0,\Re(a)>0,\Re(d)>-1$ with $\Re(\eta)>0$, $\Re(\xi)>0,\Re(\zeta)>0$, and $a,f,b,\sigma,\alpha\geq 0,f,a>\Re(c)+b$.

Proof. Whittaker transform definition, and (13)

$$\int_0^\infty y_1^{\beta-1}\exp\left(-(1/2)sy_1\right)\mathrm{W}_{\xi,\eta}(sy_1)\mathrm{J}_{d,\eta,\xi,\zeta,\alpha}^{c,a,b,f,\sigma}\left(\lambda y_1^{\vartheta}\right)$$
$$=\int_0^\infty y_1^{\beta-1}\exp\left(-(1/2)sy_1\right)\mathrm{W}_{\xi,\eta}(sy_1)\sum_{t=0}^{\infty}\frac{(\eta;\sigma,\alpha)_{at}(\zeta;\sigma,\alpha)_{ft}\left(-\lambda y_1^{\vartheta}\right)^t}{\Gamma(ct+d+1)(\xi;\sigma,\alpha)_{bt}t!}$$

inserting $sy_1=\alpha$ and applying definition of Whittaker transform, we have

$$\int_0^\infty y_1^{\beta-1}\exp\left(-(1/2)sy_1\right)\mathrm{W}_{\xi,\eta}(sy_1)J_{d,\eta,\xi,\zeta,\alpha}^{c,a,b,f,\sigma}\left(\lambda y_1^{\vartheta}\right)$$
$$=\sum_{t=0}^{\infty}\frac{(\eta;\sigma,\alpha)_{at}(\zeta;\sigma,\alpha)_{ft}(-\lambda)^t}{\Gamma(ct+d+1)(\xi;\sigma,\alpha)_{bt}t!}\frac{\Gamma((2)\pm\eta+\vartheta t+\beta)}{\Gamma(1-\xi+\vartheta t+\beta)}(s)^{-\beta-\vartheta t}$$
$$=\frac{1}{s^{\beta}}\sum_{t=0}^{\infty}\frac{(\eta;\sigma,\alpha)_{at}(\zeta;\sigma,\alpha)_{ft}}{\Gamma(ct+d+1)(\xi;\sigma,\alpha)_{bt}t!}\frac{\Gamma((2)\pm\eta+\vartheta t+\beta)\Gamma((2)\pm\eta+\beta)/\Gamma((2)\pm\eta+\beta)}{\Gamma(1-\xi+\vartheta t+\beta)\Gamma(1-\xi+\beta)/\Gamma(1-\xi+\beta)}\left(\frac{-\lambda}{s^{\vartheta}}\right)^t$$
$$=\frac{\Gamma((2)\pm\eta+\beta)}{\Gamma(1-\xi+\beta)\Gamma(d+1)}\sum_{t=0}^{\infty}\frac{(\eta;\sigma,\alpha)_{at}(\zeta;\sigma,\alpha)_{ft}((2)\pm\eta+\beta)_{\vartheta t}}{(d+1)_{ct}(\xi;\sigma,\alpha)_{bt}(1-\xi+\beta)_{\vartheta t}t!}$$
$$=\frac{\Gamma((2)\pm\eta+\beta)}{\Gamma(1-\xi+\beta)\Gamma(d+1)s^{\beta}}$$

6 Conclusion

In this paper, we investigate and present the extension of generalized (BM-IV) $J_{d,\eta,\xi,\zeta,\alpha}^{c,a,b,\sigma,\sigma}(y_1)$ function in (4) by applying a generalized Pochhammer symbol $(\rho;\varrho,\sigma)_{\alpha}$ explained in (7). Moreover, in our present investigation we discussed the Laplace transform, which is instrumental in solving differential equations and analyzing systems Whittaker transform, which plays a significant role in mathematical physics and various applied fields and Euler-Beta transform of the freshly defined function $J_{d,\eta,\xi,\zeta,\alpha}^{c,a,b,f,\sigma}(y_1)$ which is own for its applications in complex analysis and number theory.

References:

Abdalla, M., & Hidan, M. (2021). Investigation of the k-analogue of Gauss hypergeometric functions constructed by the Hadamard product. *Symmetry*, *13*(4), 714.

Abro, K. A., Atangana, A., & Gomez-Aguilar, J. F. (2021). An analytic study of bioheat transfer Pennes model via modern non-integers differential techniques. *The European Physical Journal Plus*, *136*(11), 1144. https://doi.org/10.1140/epjp/s13360-021-02136-x

Ali, R. S., Mubeen, S., Nayab, I., Araci, S., Rahman, G., & Nisar, K. S. (2020). Some fractional operators with the generalized Bessel–Maitland function. *Discrete Dynamics in Nature and Society*, *2020*, 1–15.

Andrews, G. E., Askey, R., Roy, R., Roy, R., & Askey, R. (1999). *Special functions* (Vol. 71). Cambridge university press Cambridge. https://api.taylorfrancis.com/content/chapters/edit/download?identifierName=doi&identifierValue=10.1081/E-EOE2-120009525&type=chapterpdf

Araci, S., Rahman, G., Ghaffar, A., Azeema, & Nisar, K. S. (2019). Fractional calculus of extended Mittag-Leffler function and its applications to statistical distribution. *Mathematics*, *7*(3), 248.

Arshad, M., Choi, J., Mubeen, S., Nisar, K. S., & Rahman, G. (2018). A new extension of Mittag-Leffler function. *Commun. Korean Math. Soc*, *33*(2), 549–560.

Arshad, M., Mubeen, S., Nisar, K. S., & Rahman, G. (2018). Extended Wright-Bessel function and its properties. *Communications of the Korean Mathematical Society*, *33*(1), 143–155.

Bateman, H., & Erdélyi, A. (1953). Higher transcendental functions, volume II. *Bateman Manuscript Project) Mc Graw-Hill Book Company*. https://authors.library.caltech.edu/records/cnd32-h9x80

Chaudhry, M. A., & Zubair, S. M. (2001). *On a class of incomplete gamma functions with applications*. Chapman and Hall/CRC. https://www.taylorfrancis.com/books/mono/10.1201/9781420036046/class-incomplete-gamma-functions-applications-aslam-chaudhry-syed-zubair

Choi, J., & Agarwal, P. (2013). Certain unified integrals associated with Bessel functions. *Boundary Value Problems*, *2013*(1), 95. https://doi.org/10.1186/1687-2770-2013-95

Choi, J., Parmar, R. K., & Chopra, P. (2020). Extended Mittag-Leffler function and associated fractional calculus operators. *Georgian Mathematical Journal*, *27*(2), 199–209. https://doi.org/10.1515/gmj-2019-2030

Erdélyi, A. (1956). *Asymptotic expansions*. Courier Corporation. https://books.google.com/books?hl=en&lr=&id=HMZIAwAAQBAJ&oi=fnd&pg=PP1&dq=Erd%C3%A9lyi,+A.+(1956).+Asymptotic+expansions,+Courier+Corporation,+(3).&ots=U_6_ff_bQN&sig=FF_9KrHEfJgkp108o1aehju8DD4

Ghanim, F., Al-Janaby, H. F., & Al-Momani, M. (2024). A new Euler-Beta function model with statistical implementation related to the Mittag-Leffler-Kummer function. *Kuwait Journal of Science*, *51*(1), 100106.

Ghayasuddin, M., Khan, W. A., & Araci, S. (2018). A new extension of Bessel Maitland function and its properties. *Matematicki Vesnik*, *70*(4), 292–302.

Goyal, R., Momani, S., Agarwal, P., & Rassias, M. T. (2021). An Extension of Beta Function by Using wiman's function. *Axioms*, *10*(3), 187.

Graham, R. L. (1994). *Concrete mathematics: A foundation for computer science*. Pearson Education India. https://books.google.com/books?hl=en&lr=&id=bRr2IvwTJaEC&oi=fnd&pg=PA5&dq=+Concrete+Mathematics.&ots=g2Q6SPTQYd&sig=5QfhQWY3eWsc8HOh0v3ED2Ubt10

Kobayashi, K. (1991). On Generalized Gamma Functions Occurring in Diffraction Theory *. *Journal of the Physical Society of Japan*, *60*(5), 1501–1512. https://doi.org/10.1143/JPSJ.60.1501

Kulmitra, M., & Tiwari, S. K. (2024). Investigation of Integral Transformation Associated with Extended Generalized Srivastava's Hypergeometric Multi Variable Special Function. *Mikailalsys Journal of Advanced Engineering International*, *1*(1), 57–73.

Magnus, W., Oberhettinger, F., Soni, R. P., & Wigner, E. P. (1967). Formulas and theorems for the special functions of mathematical physics. *Physics Today*, *20*(12), 81–83.

Mubeen, S., Ali, R. S., Nayab, I., Rahman, G., Abdeljawad, T., & Nisar, K. S. (2020). Integral transforms of an extended generalized multi-index Bessel function. *AIMS Mathematics*, *5*(6), 7531–7546.

Mubeen, S., Rahman, G., Nisar, K. S., Choi, J., & Arshad, M. (2017). An extended beta function and its properties. *Far East J. Math. Sci*, *102*, 1545–1557.

Olver, F. W. (2010). *NIST handbook of mathematical functions hardback and CD-ROM*. Cambridge university press. https://books.google.com/books?hl=en&lr=&id=3I15Ph1Qf38C&oi=fnd&pg=PR7&dq=Olver,+F.+W.,+Lozier,+D.+W.,+Boisvert,+R.+F.,+%26+Clark,+C.+W.+(Eds.).+(2010).+NIST+handbook+of+mathematical+functions+hardback+and+CD-ROM.+Cambridge+university+press.&ots=tMaXwDgtFX&sig=GleJY-PSVNw-LpqmhyyheBnfjbY

Paneva-Konovska, J. (2021). Series in Le Roy type functions: A set of results in the complex plane A survey. *Mathematics*, *9*(12), 1361.

Pathak, R. S. (1966). Certain convergence theorems and asymptotic properties of a generalization of Lommel and Maitland transformations. *Proc. Nat. Acad. Sci. India. Sect. A*, *36*(1), 81–86.

Petojević, A. (2008). A note about the Pochhammer symbol. *Mathematica Moravica*, *12–1*, 37–42.

Rahman, G., Agarwal, P., Mubeen, S., & Arshad, M. (2018). Fractional integral operators involving extended Mittag–Leffler function as its Kernel. *Boletín de La Sociedad Matemática Mexicana*, *24*(2), 381–392. https://doi.org/10.1007/s40590-017-0167-5

Rahman, G., Ghaffar, A., Mubeen, S., Arshad, M., & Khan, S. (2017). The composition of extended Mittag-Leffler functions with pathway integral operator. *Advances in Difference Equations*, *2017*(1), 176. https://doi.org/10.1186/s13662-017-1237-8

Rahman, G., Mubeen, S., Nisar, K. S., & Choi, J. (2019). Certain extended special functions and fractional integral and derivative operators via an extended beta function. *Nonlinear Functional Analysis and Applications*, 1–13.

Rahman, G., Nisar, K. S., Choi, J., Mubeen, S., & Arshad, M. (2019). Pathway fractional integral formulas involving extended Mittag-Leffler functions in the kernel. *Kyungpook Mathematical Journal*, *59*(1), 125–134.

Rahman, G., Saboor, A., Anjum, Z., Nisar, K. S., & Abdeljawad, T. (2020). An extension of the Mittag-Leffler function and its associated properties. *Advances in Mathematical Physics*, *2020*, 1–8.

Sachan, D. S., Rajak, B., & Ayant, F. (2024). Finite Integral Involving Incomplete Aleph-functions and Fresnel Integral. *Journal of Fractional Calculus and Applications*, *15*(2), 1–9.

Safdar, M., Rahman, G., Ullah, Z., Ghaffar, A., & Nisar, K. S. (2019). A New Extension of the Pochhammer Symbol and Its Application to Hypergeometric Functions. *International Journal of Applied and Computational Mathematics*, *5*(6), 151. https://doi.org/10.1007/s40819-019-0733-9

Sahai, V., & Verma, A. (2016). On an extension of the generalized Pochhammer symbol and its applications to hypergeometric functions. *Asian-European Journal of Mathematics*, *09*(03), 1650064. https://doi.org/10.1142/S1793557116500649

Şahin, R., & Yağcı, O. (2020). A New Generalization of Pochhammer Symbol and Its Applications. *Applied Mathematics and Nonlinear Sciences*, *5*(1), 255–266. https://doi.org/10.2478/amns.2020.1.00023

Singh, M., Khan, M. A., & Khan, A. H. (2014). On some properties of a generalization of Bessel-Maitland function. *International Journal of Mathematics Trends and Technology-IJMTT*, *14*. https://ijmttjournal.org/archive/ijmtt-v14p507

Srivastava, H. M., Çetinkaya, A., & Kıymaz, I. O. (2014). A certain generalized Pochhammer symbol and its applications to hypergeometric functions. *Applied Mathematics and Computation*, *226*, 484–491.

Srivastava, H. M., Rahman, G., & Nisar, K. S. (2019). Some Extensions of the Pochhammer Symbol and the Associated Hypergeometric Functions. *Iranian Journal of Science and Technology, Transactions A: Science*, *43*(5), 2601–2606. https://doi.org/10.1007/s40995-019-00756-8

Srivastava, R. (2013). Some generalizations of Pochhammer's symbol and their associated families of hypergeometric functions and hypergeometric polynomials. *Applied Mathematics & Information Sciences*, *7*(6), 2195.

Srivastava, R., & Cho, N. E. (2014). Some extended Pochhammer symbols and their applications involving generalized hypergeometric polynomials. *Applied Mathematics and Computation*, *234*, 277–285.

Usman, T., Khan, N., & Martínez, F. (2023). Analysis of Generalized Bessel–Maitland Function and Its Properties. *Axioms*, *12*(4), 356.

Watson, G. N. (1922). *A treatise on the theory of Bessel functions* (Vol. 2). The University Press.

Wilf, H. S. (2005). *Generatingfunctionology*. CRC press.

https://books.google.com/books?hl=en&lr=&id=bVFuBwAAQBAJ&oi=fnd&pg=PP1&dq=Wilf,+H.+S.+(2005).+generatingfunctionology.+CRC+press.&ots=u_3MYTuPMZ&sig=ism4rbGCW741ccqKRyzcEANGgaU

Wright, E. M. (1935). The asymptotic expansion of the generalized Bessel function. *Proceedings of the London Mathematical Society*, *2*(1), 257–270.

Zayed, H. M. (2021). On generalized Bessel–Maitland function. *Advances in Difference Equations*, *2021*(1), 432. https://doi.org/10.1186/s13662-021-03577-5