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On the convergence of Double Aboodh Transform

Zahida Perveen[1]*

Sadia Akram[2]

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Abstract

The double Aboodh transformation is an extension of the Aboodh transform, a mathematical technique used primarily for solving differential equations. In this research paper, we have studied the convergence properties of Double Aboodh transformation and the results have been presented in the form of theorems on convergence, absolute convergence and uniform convergence of Double Aboodh transformation. Also, Volterra Integro – Partial Differential Equation is solved by using Double Aboodh transform.

Keywords: Double Aboodh transform; Inverse Aboodh transform; Integro-Partial differential equation; Partial derivatives.

1 Introduction

In recent years, the Aboodh Transform has emerged as a significant development in integral transforms (Aboodh, 2013; Aboodh, 2014), opening new avenues for solving partial differential equations (Ahmed et al., 2023) and optimizing budget allocation (Albukhuttar et al., 2023). Its applications span various disciplines, including nonlinear oscillators (Basit et al., 2023), fractional-order systems (Ganie et al., 2024), and time fractional equations (Jani & Singh, 2023). Jasim et al. (2023) provide a comprehensive review of integral transforms, offering valuable insights into their theoretical foundations and applications. Integral transformations are helpful when working with differential equations that have specific boundary conditions. When the transformation class is chosen carefully, it is typically possible to solve an algebraic equation that can solve both the derivatives and the boundary values of an unsolvable differential equation. The

[1]Department of Mathematics, Lahore Garrison University, Pakistan
Corresponding Author: zahida.perveen@gmail.com

acquired solution is, of course, the differential equation's original solution transformed; to finish the procedure, this transformation must be inverted.

The double Laplace transform of a function $f(x,t)$ defined in the positive quadrant of the xt-plane is defined in (Dhunde & Waghmare, 2014). By using (Dhunde & Waghmare, 2014), the double Aboodh transform of a function $f(x,t)$ is defined by the equation

$$A_x A_t\{f(x,t),p,v\} = \bar{f}(p,v) = \frac{1}{p.v}\int_0^\infty e^{-px}\int_0^\infty e^{-vt} f(x,t)dtdx \qquad (1.1)$$

whenever that integral exists. Here p, s are complex numbers.

Integral transforms serve as fundamental tools in mathematical analysis, offering versatile approaches for solving differential equations and understanding complex mathematical systems. Recent research has seen a surge in the exploration of novel integral transforms and their applications across various scientific disciplines, contributing to advancements in mathematical theory and practical problem-solving methodologies.

Jasim et al. (2023) provide an extensive review of integral transforms, delving into their theoretical underpinnings, computational techniques, and practical implications. This comprehensive overview serves as a cornerstone for understanding the significance and utility of integral transforms in mathematical analysis.

Analytical investigations conducted by Liaqat et al. (2023) focus on time-fractional Black–Scholes models, employing innovative methods such as the Aboodh residual power series approach to unravel the mathematical intricacies of financial systems and derivative pricing models.

In parallel, Mansour et al. (2023) present a novel methodology for solving partial differential equations using the double complex SEE integral transform, showcasing its effectiveness in addressing complex mathematical problems and advancing computational techniques in mathematical analysis.

Pue-on (2023) explores the unique properties of the double Sadik transform and its applications to fractional Caputo partial differential equations, shedding light on its theoretical foundations and practical implications in mathematical modeling and analysis.

Moreover, Qayyum and Ahmad, (2024) introduce innovative solutions for time- and space-fractional Black–Scholes European option pricing models through the fractional extension of the He-Aboodh algorithm, contributing to the development of efficient computational techniques for financial modeling and risk management.

Saadeh's comprehensive research endeavors (2022a; 2022b; 2023a; 2023b) investigate various integral transforms and their applications, offering valuable insights into their theoretical frameworks and practical implementations. These studies contribute significantly to advancing mathematical analysis and problem-solving methodologies, enriching interdisciplinary research endeavors.

Furthermore, Sedeeg et al. (2023) explore the properties and applications of the Gamar transform, providing valuable contributions to the understanding of triple integral transforms and their potential applications in mathematical analysis and problem-solving.

In many areas of practical mathematics, engineering, and physics, non-homogeneous partial differential equations including waves, heat, and other variables can be solved with the help of this transformation. It also offers a strong technique for linear system analysis. It plays a major role in the solution of integral and differential equations. We are now going to discuss the double Aboodh transform's convergence, absolute convergence, uniform convergence, and other features. The Volterra Integro-Partial Differential Equation is solved using the double Aboodh transform in the last section (Mohseni Moghadam & Saeedi, 2010).

2 Properties of Double Aboodh integral

2.1 (a) Sufficient condition for existence:

If f (x, t) is piecewise continuous on the interval [0, ∞) and of exponential order c, d then $A_x A_t\{f(x,t)\}$ exists for s > c v > d.

Proof:

By the additive interval property of definite integral

$$A_x A_t\{f(x,t)\} = \frac{1}{s.v}\int_0^T\int_0^U e^{-st}e^{-vx}f(x,t)dxdt + \frac{1}{s.v}\int_T^\infty\int_U^\infty e^{-st}e^{-vx}f(x,t)dxdt$$
(2.1)

The integral I_1 exists. Now f is of exponential order, so there exists constant c, d M > 0, N > 0, T > 0, U > 0 so that

$$|I_2| \le \int_T^\infty\int_U^\infty \left|\frac{e^{-st}e^{-vx}}{s.v}f(x,t)\right| dxdt$$

$$= MN\frac{e^{-(s-c)T}}{s(s-c)}.\frac{e^{-(v-d)U}}{v(v-d)}$$

For s > c, v > d. so the integral $\int_T^\infty\int_U^\infty \left|\frac{e^{-st}e^{-vx}}{s.v}f(x,t)\right| dxdt$ converges by the comparison test for improper integrals. The existence of I_1 and I_2 implies that eq (2.1)

$$A_x A_t\{f(x,t)\} = \frac{1}{s.v}\int_0^T\int_0^U e^{-st}e^{-vx}f(x,t)dxdt + \frac{1}{s.v}\int_T^\infty\int_U^\infty e^{-st}e^{-vx}f(x,t)dxdt$$

exist for s > c, v > d.

2.2 (b) Behavior of F (s, v) as s, v→∞

If f (x, t) is piecewise continuous on [0, ∞) and of exponential order, then $\lim_{s,v\to\infty} A_x A_t\{f(x,t)\} = 0$.

Proof:

Since f (x, t) is piecewise continuous on [0,T] it is necessarily bounded on the interval that is $|f(x,t)| \le M_1N_1 = M_1N_1e^{0t}e^{0x}$. Also f is assumed to be of exponential order, there exist constants m, n, $M_2, N_2 > 0$ and T > 0, such that $|f(x,t)| \le M_2N_2e^{mt}e^{nx}$ for t,x > T and M,N denotes the maximum of $[M_1N_{1,}M_2N_2]$ then

$$A_xA_t\{f(x,t)\} \le \frac{1}{s.v}\int_0^\infty\int_0^\infty e^{-st}e^{-vt}|f(x,t)|\,dxdt$$
$$= \frac{M.N}{s.v(s-c)(s-d)}$$

for s >c and v > d. As s, d→∞. we have $A_xA_t\{f(x,t)\}$→0 then $\lim_{s,v\to\infty} A_xA_t\{f(x,t)\} = 0$.

2.3 (c) Region of convergence:

The region of convergence is important to understand because it defines the region where the double Aboodh transform exist.

$$A_xA_t\{f(x,t),p,v\} = \bar{f}(p,v) = \frac{1}{p.v}\int_0^\infty e^{-px}\int_0^\infty e^{-vt} f(x,t)dtdx$$

The ROC for a given f (x, t) is defined as the range for which the double Aboodh transform converges. If we consider exponential $f(x,t) = e^{-at-bx}u(x,t)$. We get the equation

$$A_xA_t\{f(x,t)\} = \frac{1}{s.v}\int_0^\infty e^{-sx-ax}\int_0^\infty e^{-vt-bt} f(x,t)dtdx$$

evaluating this we get

$$= \frac{1}{s(s+a)v(v+b)}[\lim_{t,x\to\infty} e^{-(s+a)x}.e^{-(v+b)t}]$$
$$= 0 = converges$$

3. Convergence Theorem of Double Aboodh integral

In this section, we prove the convergence theorem of (1.1)

Theorem 3 .1. Let φ(x, t) be a function of two variables continuous in the positive quadrant of the xt - plane. If the integral

$$\frac{1}{p.v}\int_0^\infty\int_0^\infty e^{-px-vt}\varphi(x,t)dxdt \qquad (3.1)$$

Converges at p = p_0, v = v_0 then integral (3.1) converges for p > p_0 , v > v_o.

For the proof we will use the following lemmas:

Lemma 3.2. If the integral

$$\frac{1}{v}\int_o^\infty e^{-vt}\varphi(x,t)dt \qquad (3.2)$$

Converges at v = v_0 then the integral (3.2) converges for v = v_o

Proof. Let

$$\alpha(x,t) = \frac{1}{v}\int_o^t e^{-v_0u}\,\varphi(x,u)du, \qquad 0 < t < \infty \qquad (3.3)$$

Clearly $\alpha(x,0) = 0$ and $\lim_{t\to\infty}\alpha(x,t)$ exist because integral $\frac{1}{v}\int_0^\infty e^{-vt}\varphi(x,t)dt$

Converges at v = v_0

By the Fundamental theorem of Calculus

$$\alpha_t(x,t) = \frac{1}{v} e^{-v_0} \varphi(x,t)$$

$$v\alpha_t(x,t)e^{v_0 t} = \varphi(x,t)$$

Choose ε_1 and R_1 so that $0 < \varepsilon_1 < R_1$

$$\frac{1}{V}\int_{\varepsilon_1}^{R_1} e^{-vt}\varphi(x,t) = \frac{1}{v}\int_{\varepsilon_1}^{R_1} e^{-vt} v e^{v_0} \alpha_t(x,t)dt$$

$$= \int_{\varepsilon_1}^{R_1} e^{-(v-v_0)t}\alpha_t(x,t)dt$$

By using integration by parts

$$=[e^{-(v-v_0)t}\alpha(x,t)]_{\varepsilon_1}^{R_1} - \int_{\varepsilon_1}^{R_1} e^{-(v-v_0)t}[-(v-v_0)]\alpha(x,t)dt$$

$$\int_{\varepsilon_1}^{R_1} e^{-vt}\varphi(x,t)dt = e^{-(v-v_0)R_1}\alpha(x,R_1) - e^{-(v-v_0)\varepsilon_1}\alpha(x,\varepsilon_1) + (v - v_0)\int_{\varepsilon_1}^{R_1} e^{-(v-v_0)}\alpha(x,t)dt$$

Now, let $\varepsilon_1 \to 0$ *Both* terms on the right which depend on ε_1 approach a limit and

$$\int_0^{R_1} e^{-vt}\varphi(x,t)dt = e^{-(v-v_o)R_1}\alpha(x,R_1) + (v-v_0)\int_0^{R_1} e^{-(v-v_0)t}\alpha(x,t)dt.$$

Now, let $R_1 \to \infty$. If $v > v_0$, the first term on the right approaches zero.

$$\int_0^{\infty} e^{-vt}\varphi(x,t)dt = (v-v_0)\int_0^{\infty} e^{-(v-v_0)t}\alpha(x,t)dt. \quad \text{For } v > v_o \qquad (3.4)$$

The theorem is proved if the integral on the right converges.

By using the Limit test for convergence [5]

For we have,

$$\lim_{t\to\infty} t^2 e^{-(v-v_0)}\alpha(x,t) = \lim_{t\to\infty}\frac{t^2}{e^{(v-v_0)t}}[\lim_{t\to\infty}\alpha(x,t)]$$

$$= 0 * [\lim_{t\to\infty}\alpha(x,t)] = 0 = \text{ finite}$$

Therefore, integral on the right of (3.4) converges for $v > v_0$

Hence the integral $\frac{1}{v}\int_0^{\infty} e^{-vt}\varphi(x,t)dt$ converges for $v > v_0$.

Lemma 3.3. If (a) integral

$$\text{h}(x, v) = \frac{1}{v}\int_0^{\infty} e^{-vt}\varphi(x,t)dt \qquad (3.5)$$

Converges for $v \geq v_0$ and (b) integral

$$\frac{1}{p}\int_0^{\infty} e^{-px}h(x,v)dx \qquad (3.6)$$

Converges for $p = p_0$ then the integral (3.6) converges for p > p_0.

Proof. Let

$$\beta(x,v) = \frac{1}{p}\int_0^x e^{-p_o u}\ h(u,v)du, \qquad o < x < \infty \tag{3.7}$$

Therefore $\beta(0,v) = 0$ and $\lim_{x\to\infty}\beta(x,v)$ exists because integral $\frac{1}{p}\int_0^\infty e^{-px}h(x,v)dx$ converges at p = p_0.

By Fundamental theorem of Calculus

From (3.7), $\beta_x(x,v) = \frac{1}{p}e^{-p_0 x}h(x,v)$.

$$p\beta_x(x,v)e^{p_0 x} = h(x,v)$$

Choose ε_2 and R_2 so that $0 < \varepsilon_2 < R_2$.

$$\frac{1}{p}\int_{\varepsilon_2}^{R_2} e^{-px}\, h(x,v)dx = \frac{1}{p}\int_{\varepsilon_2}^{R_2} e^{-px}\, pe^{-p_{0}x}\beta_x(x,v)dx$$

$$= \int_{\varepsilon_2}^{R_2} e^{-(p-p_0)x}\beta_x(x,v)dx$$

$$= \left[e^{-(p-p_0)x}\beta(x,v)\right]_{\varepsilon_3}^{R_2} - \int_{\varepsilon_2}^{R_2} e^{-(p-p_0)x}[-(p-p_0)]\beta(x,v)dx$$

$$= e^{-(p-p_0)R_2}\beta(R_2,v) - e^{-(p-p_0)\varepsilon_2}\beta(\varepsilon_2,v) + (p-p_0)\int_{\varepsilon_2}^{R_2} e^{-(p-p_0)x}\beta(x,v)dx.$$

Now let $\varepsilon_2 \to 0$. Both terms on the right which depend on ε_2 approach a limit and

$$\frac{1}{p}\int_0^{R_2} e^{-px}\, h(x,v)dx = e^{-(p-p_0)R_2}\beta(R_2,v) + (p-p_0)\int_0^{R_2} e^{-(p-p_0)x}\beta(x,v)dx$$

Now let $R_2 \to \infty$. If $p > p_0$, the first term on the right approaches zero.

$$\frac{1}{p}\int_0^\infty e^{-px}\, h(x,v)dx = (p-p_0)\int_0^\infty e^{-(p-p_0)x}\beta(x,v)dx \quad for\ p > p_0 \tag{3.8}$$

Theorem is proved if the integral on the right converges.

By using the Limit test for convergence [4].

For we have,

$$\lim_{x\to\infty} x^2 e^{-(p-p_0)x}\beta(x,v) = \lim_{x\to\infty}\frac{x^2}{e^{(p-p_0)}}\left[\lim_{x\to\infty}\beta(x,v)\right]$$

$$= 0*\left[\lim_{x\to\infty}\beta(x,v)\right] = 0 = \text{finite}.$$

Therefore, integral on the right of (3.8) converges for $p > p_0$.

Hence, the integral $\frac{1}{p}\int_0^\infty e^{-px}\, h(x,v)dx$ converges for $p > p_0$.

The proof of the theorem 3.1 is as follows.

$$\frac{1}{p.v}\int_0^\infty\int_0^\infty e^{-px-vt}\varphi(x,t)dxdt = \frac{1}{p}\int_0^\infty e^{-px}\left\{\frac{1}{v}\int_0^\infty e^{-vt}\varphi(x,t)dt\right\}dx$$

$$= \frac{1}{p}\int_0^\infty e^{-px} h(x,v)dx \qquad (3.9)$$

Where h (x, v) $= \frac{1}{v}\int_0^\infty e^{-vt}\varphi(x,t)dt.$

By Lemma 3.2 integral $\frac{1}{v}\int_0^\infty e^{-vt}\varphi(x,t)dt$ converges for $v > v_0$.

Also, by Lemma 3.3, integral $\frac{1}{p}\int_0^\infty e^{-px} h(x,v)dx$ converges for $p > p_0$.

Therefore, the integral in RHS of (3.10) converges for $p > p_0, v > v_0$.

Hence the integral

$$\frac{1}{p.v}\int_0^\infty \int_0^\infty e^{-px-vt}\varphi(x,t)dxdt$$

converges for $p > p_0, v > v_0$.

This completes the proof of the Theorem 3.1.

Corollary 3.1. If the integral (3.1) diverges at $p = p_0, v = v_0$ then the integral (3.1) diverges at $p > p_0, v > v_0$.

Corollary 3.2. The region of the convergence of the integral (3.1) is the positive quadrant of the xt -plane.

Now we prove absolute convergence of integral (3.1).

Theorem 3.2. If the integral (3.1) converges absolutely at $p = p_o, v = v_o$ then integral (3.1) converges absolutely for $p \geq p_o , v = v_o$.

Proof. We know that

$$e^{-px-vt}|\varphi(x,t)| \leq e^{-p_0 x} \qquad p_0 \leq p < \infty , v_0 \leq v < \infty$$

Therefore

$$\frac{1}{p.v}\int_0^\infty \int_0^\infty e^{-px-vt} |\varphi(x,t)|dt\mathrm{dx} \leq \frac{1}{p.v}\int_0^\infty \int_0^\infty e^{-p_0x-v_0t} |\varphi(x,t)|dt\mathrm{dx}$$

For given hypothesis

$$\frac{1}{p.v}\int_0^\infty \int_0^\infty e^{-p_0x-v_0t} |\varphi(x,t)|dtdx \text{ converges.}$$

Hence, $\frac{1}{p.v}\int_0^\infty \int_0^\infty e^{-px-vt} |\varphi(x,t)|dt\mathrm{dx}$ converges for $p \geq p_o , v = v_o$.

Therefore, integral (3.1) converges absolutely for $p \geq p_o , v = v_o$.

4. Uniform convergence

In this section we prove the uniform convergence of (1.1).

Theorem 4.1. If $f(x,t)$ is continuous on $[0,\infty) \times [0,\infty)$ and

$$H(x,t) = \frac{1}{p.v}\int_0^x \int_0^t e^{-p_0 m - v_0 n} f(m,n)dmdn \qquad (4.1)$$

Is bounded on $[0,\infty) \times [0,\infty)$, then the double Aboodh transform of f converges uniformly on [p,∞) × $[v, \infty)$ if $p > p_0, v > v_0$.

For the proof we will use the following lemmas:

Lemma 4.2. If $g(x,t) = \frac{1}{v}\int_0^t e^{-v_0 n} f(x,n)dn$ is bounded on [0,∞) then the Aboodh transform of f with respect to v converges uniformly on [v,∞) if $v > v_0$.

Proof: If $0 \leq r \leq r_1$

$$\frac{1}{v}\int_r^{r_1} e^{-vt} f(x,t)dt = \frac{1}{v}\int_r^{r_1} e^{-(v-v_0)t} e^{-v_{0t}} f(x,t)dt$$

$$= \int_r^{r_1} e^{-(v-v_0)t} g_t\,(x,t)dt.$$

Using integration by parts

$$= e^{-(v-v_0)r_1} g(x,r_1) - e^{-(v-v_0)r} g(x,r) + (v - v_o)\int_r^{r_1} e^{-(v-v_0)t} g(x,t)dt$$

Therefore if $|g(x,t)| \leq M$ then

$$\frac{1}{v}\int_r^{r_1} e^{-vt} f(x,t)dt \mid \ \leq M\{e^{-(v-v_0)r_1} + e^{-(v-v_0)r} + (v - v_0)\int_r^{r_1} e^{-(v-v_o)t} dt$$

$$= M\{e^{-(v-v_0)r_1} + e^{-(v-v_0)r} - e^{-(v-v_0)r_1} + e^{-(v-v_0)r}\}$$

$$= 2Me^{-(v-v_0)r}, \quad for \ \ v > v_0.$$

By Cauchy criterion for uniform convergence [4].

$$= \frac{1}{v}\int_r^{r_1} e^{-vt} f(x,t)dt$$

Converges uniformly on $[v, \infty)\ if\ v > v_0$.

Hence, Aboodh transform of f with respect to v converges uniformly on $[v, \infty)\ \ if\ v > v_0$.

□

Lemma 4.3. If (a) integral $g(x,\infty) = \frac{1}{v}\int_0^\infty e^{-vt} f(x,t)dt$ convergence uniformly on $[v, \infty)$ if $v > v_0$, (b) $\alpha(x,v) = \frac{1}{p}\int_0^x e^{-p_0 u} g(u\,,v)du$ is bounded on $[0,\infty)$ then the Aboodh transform of f with respect to v converges uniformly on $[p,\infty)$ if $p > p_0$.

Proof: Proof is similar to Lemma 4.2.

The proof of the theorem 4.1 is as follows:

$$H(x,t) = \frac{1}{p.v}\int_0^x \int_0^t e^{-p_0 m - v_0 n} f(m,n)dmdn$$

$$= \frac{1}{p}\int_0^x e^{-p_0 m}\left[\frac{1}{v}\int_0^t e^{-v_0 n} f(m,n)dn\right] dm$$

$$= \frac{1}{p} \int_0^x e^{-p_0 m} g(m,t) dm$$

Where $g(m,t) = \frac{1}{v} \int_o^t e^{-v_0 n} f(m,n) dn$ is bounded on $[0,\infty)$.

By Lemma 4.2, Aboodh transform of f with respect to v converges uniformly on $[v,\infty)$ if $v > v_0$.

Also, by Lemma 4.3, Aboodh transform of g with respect to p converges uniformly on $[p,\infty)$ if $p > p_0$.

Hence double Aboodh transform of f converges uniformly on $[p,\infty) \times [v,\infty)$

If $p > p_0, v > v_0$. ¤

We now prove the differentiability of double Aboodh transform

Theorem 4.4. If $f(x,t)$ is continuous on $[0,\infty) \times [0,\infty)$ and

$$H(x,t) = \frac{1}{p.v} \int_0^x \int_0^t e^{-p_o m - v_0 n} f(m,n) dm dn$$

is bounded on $[0,\infty) \times [0,\infty)$ then the double Aboodh transform of f is infinitely differentiabile with respect to p and v on $[p,\infty), [v,\infty)$ if $p > p_0$, $v > v_0$ with

$$\frac{\partial^{y+z}}{\partial p^y \partial v^z} \bar{f}(o,v) = (-1)^{y+z} \frac{1}{p.v} \int_o^\infty \int_0^\infty e^{-px-vt} x^y t^z f(x,t) dt dx. \quad (4.2)$$

For the proof we will use the following lemmas:

Lemma 4.5. If $g(x,t) = \frac{1}{v} \int_0^t e^{-v_0 n} f(x,n) dn$ is bounded on $[0,\infty)$ then the Aboodh transform of f is infinitely differentiable with respect to v on $[\ ,\infty)$ if $v > v_0$, with

$$\frac{\partial^z}{\partial s^z} \bar{f}(x,v) = (-1)^z \frac{1}{v} \int_0^\infty e^{-vt} t^z f(x,t) dt \quad (4.3)$$

Proof. First, we prove that the integrals

$$I_z(x,v) = (-1)^z \frac{1}{v} \int_0^\infty e^{-vt} t^z f(x,t) dt \qquad z = 0,1,2,3 \ldots ..$$

All converges uniformly on $v,\infty)$ if $v > v_0$.

If $0 < r < r_1$, then

$$\frac{1}{v} \int_r^{r_1} e^{-vt} t^z f(x,t) dt = \frac{1}{v} \int_r^{r_1} e^{-(v-v_0)t} t^z g_x(x.t) dt$$
$$= \frac{1}{v} [e^{-(v-v_0) r_1} r_1^z g(x,r_1) - e^{-(v-v_0) r} g(x,r)$$
$$- \int_r^{r_1} \left\{ \frac{d}{dt} \left[e^{-(v-v_0)t} t^z \right] \right\} g(x,t) dt].$$

Therefore, if $|g(x,t)| \leq M < \infty$ on $[0,\infty)$ then

$$\left| \int_r^{r_1} e^{-vt} t^z f(x,t) dt \right| \leq M \left\{ e^{-(v-v_0) r} r^z + e^{-(v-v_0) r} r^z + \int_0^\infty \left\{ \frac{d}{dt} e^{-(v-v_0)t} t^z \right\} dt \right\}$$

Therefore, since $e^{-(v-v_0)r}r^z$ decrease monotonically on (v,∞) if $v > v_0$

. $\left|\frac{1}{v}\int_r^{r_1} e^{-vt}t^z f(x,t)dt\right| \le 3Me^{-(v-v_0)r}r^z$, $\qquad 0 < r < r_1$.

By Cauchy criterion for uniform convergence [4].

$I_n(x,v)$ converges uniformly on $[v,\infty)\ if\ v > v_0$.

Now using [4, pp.18-19] and induction proof we have (4.3).

That is using Aboodh transform of f is infinitely differentiable with respect to v on $[v,\infty)\ if\ v > v_0$

Lemma 4.6. If (a) the integral $\varphi(x,v) = \frac{1}{v}\int_0^\infty e^{-vt}t^z f(x,t)dt$ converges uniformly on $[v,\infty)\ \ if\ \ v > v_0$.(b) $h(x,v)\frac{1}{p}\int_0^x e^{-p_0 m}\varphi(x,v)dx$ is bounded on $[0,\infty)$ then the Aboodh transform of φ is infinitely differentiable with respect to p on [, ∞) if $p > p_0$ with

$$\frac{\partial^y}{\partial p^y}\varphi(x,v) = (-1)^y\frac{1}{v}\int_0^\infty e^{-vt}t^y\,\varphi(x,v)dx. \qquad (4.4)$$

Proof. Proof is similar to Lemma 4.5.

The proof of the theorem 4.4 is as follows

$$H(x,t) = \frac{1}{p.v}\int_0^x\int_0^t e^{-p_0 m - v_0 n}\,f(m,n)dmdn.$$

$$= \frac{1}{p}\int_0^\infty e^{-p_0 m}\left\{\frac{1}{v}\int_0^t e^{-v_0 n}f(m,n)dn\right\}dm$$

$$= \frac{1}{p}\int_0^x e^{-p_0 m}g(m,t)dm$$

Where $g(m,t) = \frac{1}{v}\int_0^t e^{-v_0 n}f(m,n)dn$ is bounded on $[0,\infty)$

By Lemma 4.5 Aboodh transform of f is infinitely differentiable with respect to v on $[v,\infty)\ if\ v > v_0$.

Also, by Lemma (4.6) Aboodh transform of g is infinitely differentiable with respect to p on $[p,\infty) if\ p > p_0$.

Hence double Aboodh transform of f is infinitely differentiable with respect to p and v on $[p,\infty)\times[v,\infty)\ \ if\ p > p_0\ ,\ \ v > v_0$,

5. Double Aboodh Transform of Double Integral

We now find double Aboodh transform of double integral.

Theorem 5.1. If $A_x A_t\{f(x,t)\} = \bar{f}\,(p,v)$ and

$$g(x,t) = \int_0^x \int_0^t f(m,n)dmdn \quad (5.1)$$

then

$$A_x A_t \left\{\frac{1}{p.v} \int_o^x \int_0^t f(m,n)dmdn\right\} = \frac{\bar{f}(p,v)}{p.v} \quad (5.2)$$

Proof. Denote $h(x,t) = \int_o^t f(x,n)dn$

By fundamental theorem of calculus

$$h_t(x,t) = f(x,t) \quad (5.3)$$

And $h(x,0) = 0$ (5.4)

Taking double Aboodh transform of equation (5.3) we get

$$v\bar{h}(p'v) - \frac{1}{v}\bar{h}(p,0) = \bar{f}\,(\text{p, v})$$

(5.5)

And single Aboodh transform of equation (5.4)

$$\bar{h}(\text{p, 0}) = 0$$

Then equation (5.5) becomes

$$\overline{h}\,(p,v) = \frac{\bar{f}(p,v)}{v}$$

(5.6)

From (5.1), $g(x,t) = \int_0^x h(u,t)du$

$$g_x(x,t) = h(x,t) \text{ and } g(o,t) = 0,$$

$$p\bar{g}(p,v) - \frac{1}{u}\bar{g}(o,v) = \bar{h}(p,v).$$

Now by using (5.6) and (5.1), we obtain

$$A_x A_t \left\{\frac{1}{p.v} \int_o^x \int_0^t f(m,n)dmdn\right\} = \frac{\bar{f}(p,v)}{p.v}$$

6 Application of Double Aboodh Transform

Here we use the double Aboodh transform to solve Volterra Integro-Partial Differential Equation which is already solved in [7] using differential transform method.

Example 6.1. Consider the following Volterra Integro partial Differential Equation:

$$\frac{\partial u(x,y)}{\partial x} + \frac{\partial u(x,y)}{\partial y} = -1 + e^x + e^y + e^{x+y} + \int_0^x \int_0^y u(r,t)drdt \quad (6.1)$$

Subject to the initial conditions:

$$\text{u(x, 0)} = e^x \; and \; u(0,y) = e^y. \quad (6.2)$$

Applying double Aboodh transform of equation (6.1), we get

$$p\bar{u}(p,u) - \frac{1}{p}\bar{u}(0,v) + v\bar{u}(p,v) - \frac{1}{v}\bar{u}(p,0)$$

$$= \frac{-1}{p^2v^2} + \frac{1}{pv^2(p-1)} + \frac{1}{vp^2(v-1)} + \frac{1}{pv(p-1)(v-1)} + \frac{\bar{u}(p,v)}{pv} \tag{6.3}$$

And single Aboodh transform of equation (6.2), we get

$$\bar{u}\,(p,0) = \frac{1}{p(p-1)} \text{ and } \bar{u}(0,v) = \frac{1}{v(v-1)} \tag{6.4}$$

Substitution (6.4) in (6.3) and simplifying, we obtain

$$\bar{u}(p,v) = \frac{1}{p(p-1)v(v-1)}$$

By using double inverse Aboodh transform, we obtain solution of (6.1) as follows:

$$u(x,t) = e^{x+t}.$$

7. Conclusion

The Aboodh Transform has emerged as a significant development in integral transforms in opening new avenues for solving partial differential equations. The features of the double Aboodh transform, including convergence, absolute convergence, and uniform convergence, were discussed in this study. In addition to this, we were able to derive the double Aboodh transform of the double integral, which we then applied to the Volterra integro-partial differential equation.

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