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Dynamical Study of Time Fractional Non Linear Wave-Like and Fisher's Equations Using Approximate Technique

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Abstract

The Homotopy Perturbation Method (HPM) is a powerful and dynamic technique for solving both linear and nonlinear partial differential equations. Since obtaining an exact solution for a nonlinear partial differential equation is challenging, perturbation approaches become valuable based on their criteria. The Homotopy Perturbation Method offers an approximate solution by utilizing the given conditions. It is worth noting that only a few terms are needed to achieve a highly accurate approximate solution. In this paper, we have employed this method and obtained the most accurate result by considering only four terms. The graphical representation of the result illustrates the precise physical situation and the accuracy of the solution. The HPM enables us to find the solution of nonlinear fractional order partial differential equations in the form of a series with easily computable components. Through the calculations and graphical representation, it becomes evident how the solution of the original equation and its behavior depends on the initial conditions.

Keywords: Fisher's Equation, Wave-Like Equation, Initial Condition, Approximate Solution, Approximate Technique. Fractional Calculus.

1 Introduction

In 1998, Huan introduced the Homotopy Perturbation Method (HPM). Recently, this

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method has gained popularity and recognition among researchers due to its simplicity and its ability to provide highly effective solutions for complex problems in various scientific and technological fields. Many mathematical models describing physical phenomena involve partial differential equations, which are used to simplify the representation of reality. The behavior of these models depends on specific input data, such as boundary or initial conditions, coefficient functions, and forcing functions. These input data give rise to localized properties of the model's solution in space and time. Therefore, studying exact or approximate solutions helps us understand the nature of these mathematical models and their real-world implications, often visualized through graphical representations. The main objective of this paper is to apply the Homotopy Perturbation Method (HPM) to find approximate solutions with fractional derivatives of nonlinear partial differential equations with given initial conditions (He, 1999; He, 2000; He, 2003; He, 2005a; He, 2005b; He, 2006; He, 2008).

In recent years, solving nonlinear differential equations has become crucial for modeling complex phenomena in various scientific fields. Techniques like the HPM and the Variational Iteration Method (VIM) have been extensively utilized. For instance, HPM has proven effective for nonlinear parabolic equations with nonlocal boundary conditions (Ghoreishi & Md. Ismail, 2011), while VIM has been used for nonlinear reaction-diffusion-convection problems (Duangpithak & Torvattanabun, 2012). Additionally, the modified cubic B-spline collocation method has provided solutions for Fisher’s reaction-diffusion equation. Mustahsan et al. have developed an efficient analytical technique specifically designed to address time-fractional parabolic partial differential equations, which are integral in various fields of science and engineering. Their study presents significant improvements in the accuracy and computational efficiency of solving these equations (Mustahsan et al., 2020). Additionally, Mustahsan et al. explored the thermo-physical behavior of metallic porous fins under varying convective loads, providing valuable insights into the performance and optimization of thermal systems (Mustahsan et al., 2021).

Fractional-order equations are also gaining attention due to their ability to describe physical processes more accurately. Research has explored the dynamics of time fractional order Φ4 equations (Younas, 2022) and the analysis of COVID-19 dynamics using fractional models (Younas, 2021). The Natural Transform method and the Optimal Homotopy Asymptotic Method (OHAM) are notable techniques for addressing these equations, showing efficacy in various applications (Nadeem, 2022; Younas, 2019).

Further advancements include OHAM for fractional-order Fredholm integro-differential equations (Iqbal et al., 2015) and its application to heat and wave-like partial differential equations (Sarwar et al., 2015). These methods continue to evolve, offering robust tools for solving complex nonlinear systems.

The perturbation technique, which is an approximate method for solving nonlinear differential equations, is widely used by engineers to tackle practical problems, often yielding interesting and important results (Mishra et al., 2023). However, perturbation methods have inherent limitations. Firstly, they rely on small or large parameters, requiring at least one unknown to be expressed as a series of small parameters (Ramya et al., 2024). Unfortunately, not all nonlinear differential equations possess such small parameters. Secondly, even if such a parameter exists, perturbation methods generally produce valid results only for small parameter values. Additionally, simplified linear equations obtained through perturbation techniques may exhibit different properties than the original nonlinear differential equation, and certain initial or boundary conditions may become irrelevant (Ahmed et al., 2024) . Consequently, initial approximations derived from these simplified equations may deviate significantly from the exact solutions. These limitations stem from the assumption of small parameters in perturbation techniques (Ahmad et al., 2021). Therefore, there is a need for the development of a new nonlinear analytical method that does not depend on small parameters (Noor et al., 2024) . Further Ahmad et al. analyzed dispersive optical solitons in nonlinear models using an analytical technique and its applications (Ahmad et al. 2023). Ji Huan He, proposed a nonlinear analytical technique that overcomes the requirement for small parameters, enabling the solution of nonlinear problems without relying on their presence. This method is based on homotopy, an important concept in topology (Shahid et al., 2024) . By utilizing a particular property of homotopy, any nonlinear problem can be transformed into an infinite number of linear problems, regardless of the existence of small or large parameters (Yasin et al., 2024).

To illustrate the general procedure, let's consider a nonlinear partial differential equation of the form:

$$\mathrm{V_t = F(s, t, v, v_s, v_{ss})\ s, t) \in (a, b) \times (0, T)}$$

With the initial condition:

$$V(s,0) = v_0 = f(s)\ s \in (a,b)$$

Here, f is a function of variables, and F is a function of differential operators and variables. Several approximate analytical schemes can be used to solve such operator equations, including the Adomian Decomposition Method (ADM), Variation Iteration Method (VIM), Homotopy Analysis Method (HAM), and tan h-expansion method (Naeem et al., 2021).

These schemes provide infinite series solutions and avoid the problem of rounding errors. Applying these methods demonstrates their applicability, accuracy, and efficiency in solving a wide range of nonlinear equations in physics, engineering, and various branches of mathematics. The structure of our paper is as follows:

Section 2, preliminary, presents the basic definitions of fractional calculus, (i) Riemann Liouville for Fractional integration and (ii) Caputo fractional derivative. In Section 3, we apply HPM to obtain analytical approximate for two different types of fractional order partial differential equations with initial conditions(Ghosh & Maitra, 2021) . In Section 4, we compare the graphical representations of each solution to the corresponding approximate solutions obtained, highlighting the accuracy of our approach. Finally, in Section 5, we provide our concluding remarks.

2 Preliminaries

This section presents several fundamental definitions that play a important role in the study of fractional calculus theory (Ullah et al., 2024) . Over the past two centuries, numerous definitions for fractional derivatives and integrals have been proposed by prominent mathematicians such as Riemann, Liouville, and Caputo. For a detailed exploration of the mathematical properties of fractional derivatives and integrals, one may refer to the works of (He, 1999; He, 2000; He, 2003; He, 2005a; He, 2005b)

2.1 Basic Definitions of Fractional Calculus

A real function $h(s), s > 0$, is said to be in space $C_\mu, \mu \in \Re$, if there exists a real number $p > \mu$ such that $h(s) = s^p h_1(s)$, where $h_1(s) \in C(0, \infty)$, and it is said to be in the space C_μ^m if and only if $h^m \in C_\mu, m \in N$.

2.2 Definition 1

The Riemann - Liouville. Fractional integral operator of order $\alpha > 0$ of a function $h \in C_\mu, \mu \geq -1$ is defined as

$$\mathrm{I}_{\mathrm{a,x}}^{\alpha}\mathrm{h}(s) = \frac{1}{\Gamma(\alpha)}\int_a^x (s-\mu)^{\alpha-1}\,\mathrm{h}(\mu)\mathrm{d}\mu,\ \alpha > 0, \mathrm{x} > 0$$

2.3 Definition 2

The fractional derivative of $f(s)$ in Caputo sense is defined as

$$\mathrm{D}_{\mathrm{a,s}}^{\alpha}\mathrm{h(s)} = \frac{1}{\Gamma(\mathrm{m}-\alpha)}\int_a^s (s-\mu)^{\mathrm{m}-\alpha-1}\,\mathrm{h}(\mu)\mathrm{d}\mu$$
$$m-1 < \alpha < m, m \in \mathbb{Z}^+, \mathrm{s} > 0.$$

3 Applications of Non-Linear Real-life Model

In this section, we have applied the HPM to obtain the analytical approximate solutions of these Non-Linear fractional order partial differential equations with the different initial conditions.

3.1 Application

Let's examine the wave-like equation [10] expressed in the following form.

$$\frac{\partial^\alpha \mathrm{v}}{\partial \mathrm{t}^\alpha} = \frac{\partial^2 \mathrm{v}}{\partial \mathrm{s}^2} + \mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{s}} + \mathrm{v} - \mathrm{v}^2$$

Now let us try to find the solution using HPM with the following initial condition:

$$v(s,0) = 1 + e^s$$

Equation (4) is a nonlinear partial differential equation. So, in principle, to get an exact solution is very difficult. However, here we can see that

$$v(s,t) = 1 + e^{s+t}$$

is an exact solution to the given Problem.

Solution

Now, let us try to solve this equation using HPM In order to solve equation (4) using HPM, the Power series can be constructed as follows

$$H(v,p) = (1-p)\left(\frac{\partial^\alpha v}{\partial t^\alpha} - \frac{\partial^\alpha v_0}{\partial t^\alpha}\right) + p\left[\frac{\partial^\alpha v}{\partial t^\alpha} - \frac{\partial^2 v}{\partial s^2} - v\frac{\partial v}{\partial s} - v + v^2\right] = 0, p \in [0,1]$$

Now for $p = 0$ and $p = 1$ equation (6), we will have the following form

$$H(v,0) = \frac{\partial^\alpha v}{\partial t^\alpha} - \frac{\partial^\alpha v_0}{\partial t^\alpha} = 0$$

$$H(v,1) = \frac{\partial^\alpha v}{\partial t^\alpha} - \frac{\partial^2 v}{\partial s^2} - v\frac{\partial v}{\partial s} - v + v^2 = 0$$

After simplification, we have the following equation

$$\left(\frac{\partial^\alpha v}{\partial t^\alpha}\right) - p\left[\frac{\partial^2 v}{\partial s^2} + v\frac{\partial v}{\partial s} + v - v^2\right] = 0$$

Suppose the solution of (4) has the form

$$\mathrm{v} = \sum_{\mathrm{i}=0}^{\infty} \mathrm{v_i p^i}$$

Substituting (10) into equation (9) yields

$$\frac{\partial^\alpha(\mathrm{v_0P^0 + v_1P^1} + \cdots)}{\partial \mathrm{t}^\alpha} + \mathrm{P}\left[\frac{\partial^2(\mathrm{v_0P^0 + v_1P^1} + \cdots)}{\partial \mathrm{s}^2} + \mathrm{v}\frac{\partial(\mathrm{v_0P^0 + v_1P^1} + \cdots)}{\partial \mathrm{s}} + (\mathrm{v_0P^0 + v_1P^1} + \cdots) - (\mathrm{v_0P^0 + v_1P^1} + \cdots)^2\right] = 0$$

Comparing the coefficient of terms with identical powers of P leads to:

$$\mathrm{P^0}: \frac{\partial^\alpha \mathrm{v_0}}{\partial \mathrm{t}^\alpha} = 0$$

$$P^1: \frac{\partial^\alpha v_1}{\partial t^\alpha} - \left[\frac{\partial^2 v_0}{\partial s^2} + v_0 \frac{\partial v_0}{\partial s} + v_0 - v_0^2\right] = 0$$
$$P^2: \frac{\partial^\alpha v}{\partial t^\alpha} - \left[\frac{\partial^2 v_1}{\partial s^2} + v_0 \frac{\partial v_1}{\partial s} + v_1 \frac{\partial v_0}{\partial s} + v_1 - 2v_0 v_1\right] = 0$$
$$P^3: \frac{\partial^\alpha v_3}{\partial t^\alpha} - \left[\frac{\partial^2 v_2}{\partial s^2} + v_0 \frac{\partial v_2}{\partial s} + v_2 \frac{\partial v_0}{\partial s} + v_1 \frac{\partial v_1}{\partial s} + v_2 - 2v_0 v_2 - v_1^2\right] = 0$$

Subsequently solving the above equations, we have

$$v_0(s,t) = 1 + e^s$$
$$v_1(s,t) = e^s \frac{t^\alpha}{\alpha!}$$
$$v_2(s,t) = e^s \frac{t^{2\alpha}}{2\alpha!}$$
$$v_3(s,t) = e^s \frac{t^{3\alpha}}{3\alpha!}$$

Finally, the approximate solution in a series form,

$$v(s,t) = v_0 + v_1 + v_2 + v_3$$
$$v(s,t) = 1 + e^s \left[1 + \frac{t^\alpha}{\alpha!} + \frac{t^{2\alpha}}{2\alpha!} + \frac{t^{3\alpha}}{3\alpha!}\right]$$

3.2 Application

To apply HPM in more complicated problem let us consider Fisher's equation

$$\frac{\partial^\alpha v}{\partial t^\alpha} = \frac{\partial^2 v}{\partial s^2} + 6v(1 - v)$$

And try to find the solution with the following initial condition

$$v(s,0) = -\frac{1}{4}\left[\operatorname{sech}^2\left(-\sqrt{\frac{1}{4}}s\right) - 2\tanh\left(-\sqrt{\frac{1}{4}}s\right) - 2\right]$$

The exact solution of the Problem is given as

$$v(s,t) = -\frac{1}{4}\left[\operatorname{sech}^2\left(-\sqrt{\frac{1}{4c}}s + \frac{5}{2}t^\alpha\right) - 2\tanh\left(-\sqrt{\frac{1}{4c}}s + \frac{5}{2}t^\alpha\right) - 2\right]$$

Solution

In order to solve equation (22) using HPM, Power series can be Constructed as follows

$$H(v,p) = (1-p)\left(\frac{\partial^\alpha v}{\partial t^\alpha} - \frac{\partial^\alpha v_0}{\partial t^\alpha}\right) + p\left[\frac{\partial^\alpha v}{\partial t^\alpha} - \frac{\partial^2 v}{\partial s^2} - 6v(1-v)\right] = 0, p \in [0,1]$$

Now for $\mathrm{p} = 0$ and $\mathrm{p} = 1$ equation (25), we will have the following form

$$H(v,0) = \frac{\partial^\alpha v}{\partial t^\alpha} - \frac{\partial^\alpha v_0}{\partial t^\alpha} = 0$$

$$H(v,1) = \frac{\partial^\alpha v}{\partial t^\alpha} - \frac{\partial^2 v}{\partial s^2} - 6v(1-v) = 0$$

After simplification, we have the following equation

$$\left(\frac{\partial^\alpha v}{\partial \mathrm{t}^\alpha}\right) - \mathrm{p}\left[\frac{\partial^2 \mathrm{v}}{\partial \mathrm{s}^2} + 6\mathrm{v}(1-\mathrm{v})\right] = 0$$

Suppose the solution of (22) has the form

$$\mathrm{v} = \sum_{\mathrm{i}=0}^{\infty} \mathrm{v_i p^i}$$

Substituting (29) into equation (28) yields

$$\frac{\partial^\alpha (\mathrm{v_0 P^0 + v_1 P^1} + \cdots)}{\partial \mathrm{t}^\alpha} - \mathrm{P}\left[\frac{\partial^2 (\mathrm{v_0 P^0 + v_1 P^1} + \cdots)}{\partial \mathrm{s}^2} + 6(\mathrm{v_0 P^0 + v_1 P^1} + \cdots)(1 - (\mathrm{v_0 P^0 + v_1 P^1} + \cdots)\right] = 0$$

Comparing the coefficient of terms with identical powers of p leads to:

$$\mathrm{P}^0: \frac{\partial^\alpha \mathrm{v}}{\partial \mathrm{t}^\alpha} = 0$$

$$\mathrm{P}^1: \frac{\partial^\alpha \mathrm{v}}{\partial \mathrm{t}^\alpha} - \left[\frac{\partial^2 \mathrm{v_0}}{\partial \mathrm{s}^2} + 6\mathrm{v_0} - 6\mathrm{v_0}^{\,2}\right] = 0$$

$$\mathrm{P}^2: \frac{\partial^\alpha \mathrm{v_2}}{\partial \mathrm{t}^\alpha} - \left[\frac{\partial^2 \mathrm{v_1}}{\partial \mathrm{s}^2} + 6\mathrm{v_1} + 12\mathrm{v_0 v_1}\right] = 0$$

$$\mathrm{P}^3: \frac{\partial^\alpha \mathrm{v_3}}{\partial \mathrm{t}^\alpha} - \left[\frac{\partial^2 \mathrm{v_2}}{\partial \mathrm{s}^2} + 6\mathrm{v_2} + 6\mathrm{v_1^2}\right] = 0$$

Subsequently solving the above equations, we have

$$v_0(s,t) = -\frac{1}{4}\left[\operatorname{sech}^2\left(-\sqrt{\frac{1}{4c}}s\right) - 2\tanh\left(-\sqrt{\frac{1}{4c}}s\right) - 2\right]$$
$$v_1(s,t) = \frac{10e^s t^{\alpha}}{(1+e^s)^3\alpha!}$$
$$v_2(s,t) = \frac{5e^s\big(1+5e^s(1+2e^s)\big)t^{2\alpha}}{(1+e^s)^5\alpha!}$$
$$v_3(s,t) = \frac{5e^s\left(7+e^s\left(169+e^s\big(267+25e^s(1+4e^s)\big)\right)\right)t^{3\alpha}}{3(1+e^s)^7\alpha!}$$

Finally, the approximate solution in a series is from

$$v(s,t) = v_0 + v_1 + v_2 + v_3$$
$$v(s,t) = \frac{1}{384}e^{-s}\operatorname{Sech}\left[\frac{s}{2}\right]^2 (96 + \frac{5t^{\alpha}\left(96 + \frac{24t^{\alpha}(5+11\operatorname{Cosh}[s]+9\operatorname{Sinh}[s])}{1+\operatorname{Cosh}[s]} + t^{2\alpha}\operatorname{Sech}\left[\frac{s}{2}\right]^4 (267+194\operatorname{Cosh}[s]+107\operatorname{Cosh}[2s] - 144\operatorname{Sinh}[s] + 93\operatorname{Sinh}[2s])\right)\left(1+\operatorname{Tanh}\left[\frac{s}{2}\right]\right)}{\operatorname{Gamma}[1+\alpha]}$$

4 Results and Discussion

To get a clear idea of the solution, if we plot some graphs of the application # 1 with multiple values, t-values, and set $\alpha = 0.80, \alpha = 0.90, \alpha = 1$, we get a three-dimensional image. Similarly, for a given t-value and some range of s-values. We get 2-D graphs. Some graphs obtained by solving the example 1 are explained below.

4.1 Graphical Representation of the Solution of application # 1

If we sketch a few graphs using a variety of s values and t values, we get three-dimensional graphs that help us understand the problem clearly. Similar to this, we obtain twodimensional graphs for specific t values and specific ranges of s values. The graphs that were produced from the solution are shown in some of their forms below.

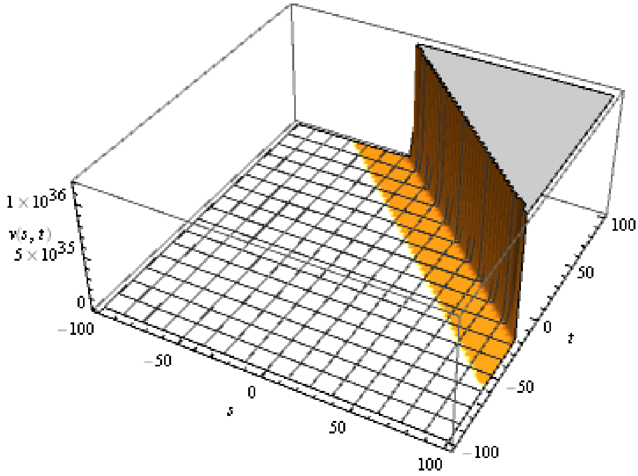


Fig: 1a

The 3D behavior of v(s,t)for α=1, t∈(−100,100),x∈(−100,100)

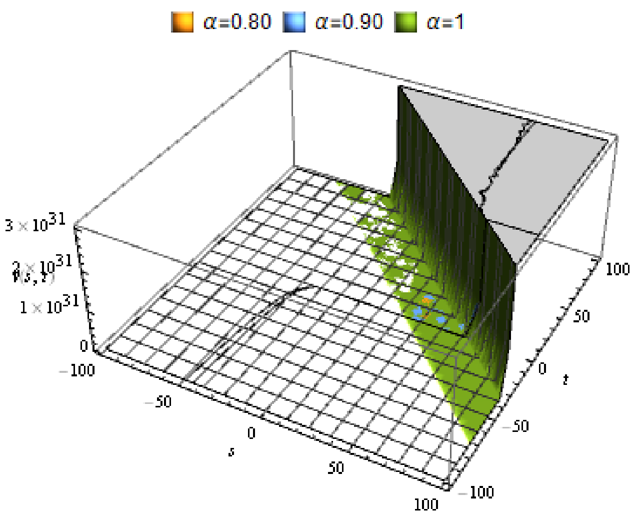


Fig: 1b

Corresponding fractional graph for different values of α and t∈(−100,100), x∈(−100,100)

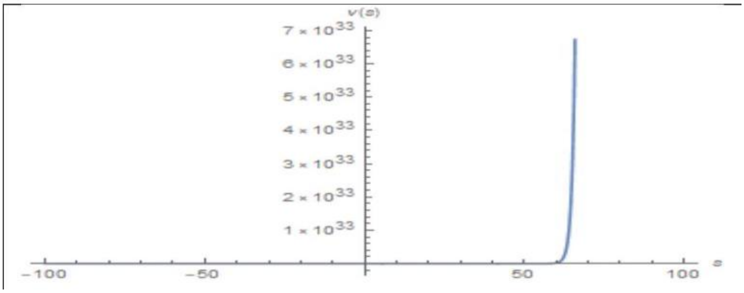


Fig: 2a

Corresponding 2D figure for $t = 100$ and $\alpha = 1$

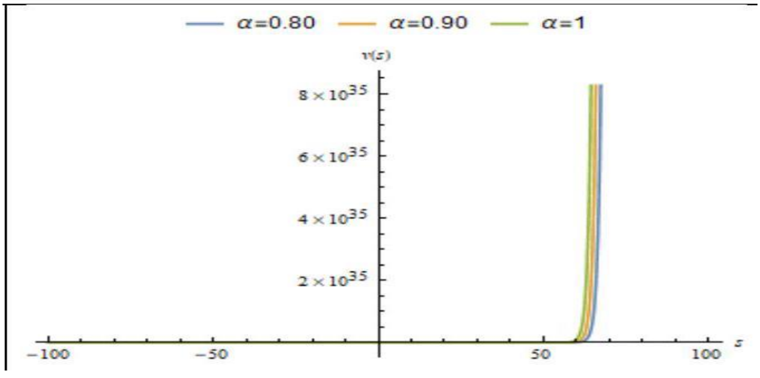


Fig: $2b$

Corresponding 2D figure for different values of α and $t = 100$

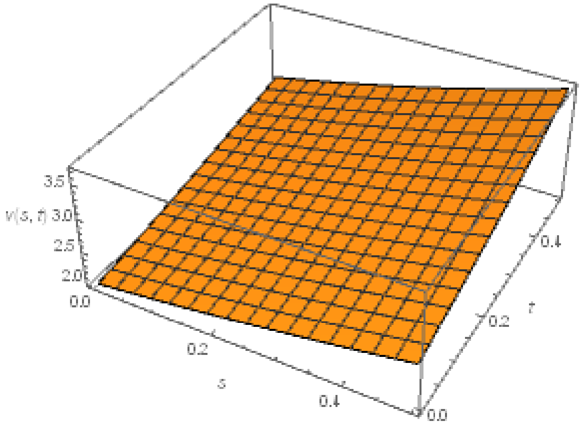


Fig: 3a

The surface of v(s,t)for α=1, t∈(−0.5,0.5),x∈(−0.5,0.5)

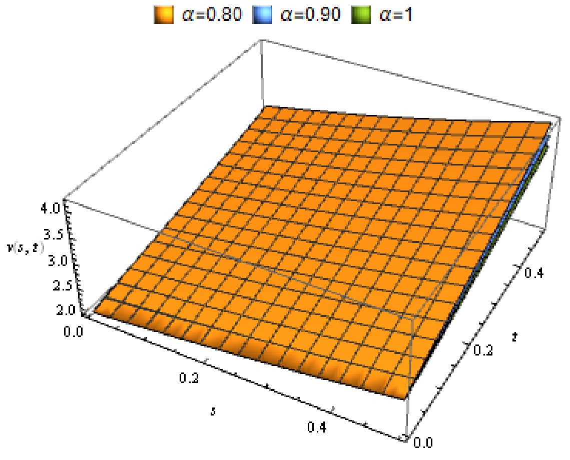


Fig: 3b

Corresponding fractional graph for different values of α, The surface of v(s,t) for

t∈(−0.5,0.5),x∈(−0.5,0.5)

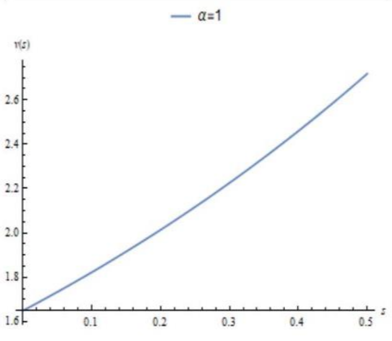


Fig: 4a

Corresponding 2D figure for t=0.5 and α = 1

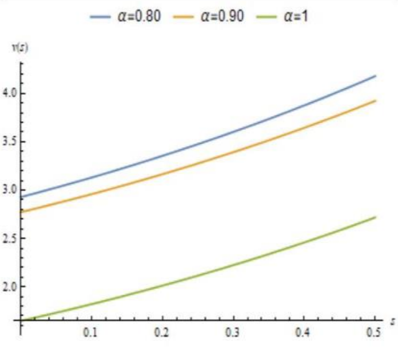


Fig: 4b

Corresponding 2D figure for different values of α and t=0.5

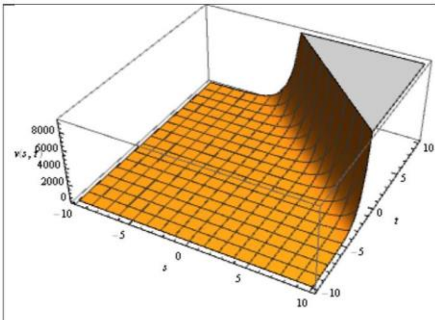


Fig: 5a

The surface of v(s, t)for $\alpha = 1$,

$t \in (-10,10), x \in (-10,10)$

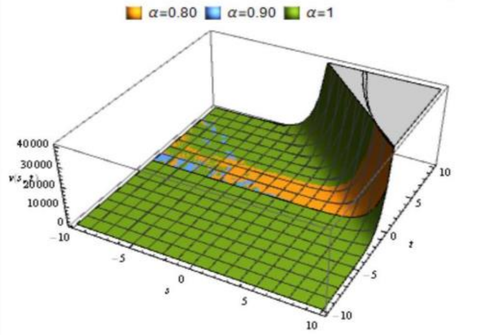
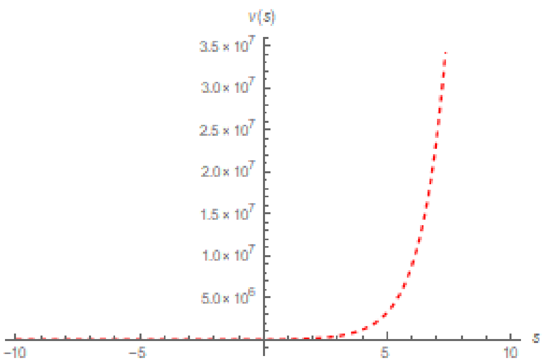


Fig: 5b

Corresponding fractional graph for different values of α and $t \in (-10,10), x \in (-10,10)$



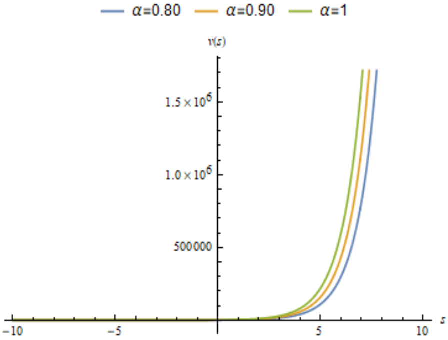


Fig:6a Corresponding 2D figure for t=10 and α=1

Fig:6b

Corresponding fractional graph for different values of α,

Corresponding 2D figure for t=10

4.2 Discussion for application # 1

The graphs show the results of solving the given differential equation for various s and t

values, and the related (two-dimensional and three-dimensional) figures are displayed in Figures (1a, 1b, 2a, 2b, 3a, 3b, 4a, 4b, 5a, 5b, 6a and 6b). We can observe that this approximation matches the exact solution exactly and also observe the fractional values $\alpha = 0.80, \alpha = 0.90$ that approaches to classical value $(\alpha = 1)$ as a result, we may draw the conclusion that this HPM is an effective technique for obtaining the nonlinear differential Equation's most precise result. However, in these instances where the approximate answer does not match the exact solution completely, the graphical depiction of the data shows how accuracy is maintained. The distinction between a precise and an approximate solution. To comprehend this circumstance, we are showing a number of the solution's graphs.

4.3 Graphical Representation of the Solution of application # 2

Three-dimensional and corresponding two-dimensional graphs represent the obtained results graphically. The two-dimensional figures are drawn for a fixed value of t, whereas the surfaces are drawn for various ranges of variables.

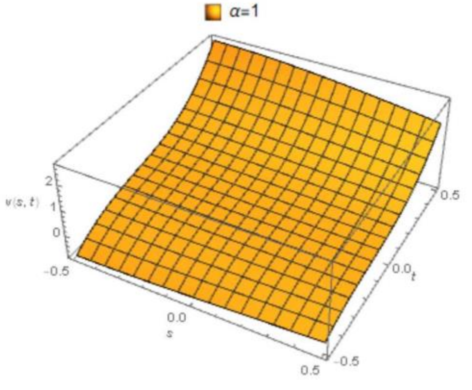


Fig:7a

The surface of $v(s,t)$ for $\alpha = 1$, $t \in (-0.5, 0.5), x \in (-0.5, 0.5)$

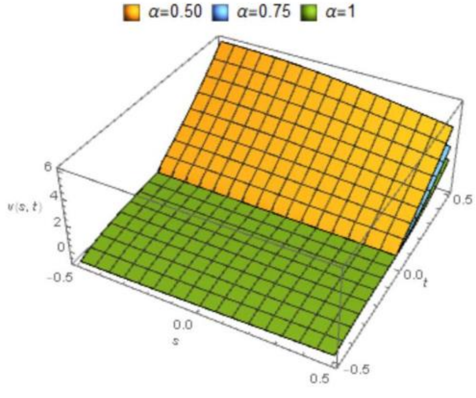


Fig:7b

Corresponding fractional graph for different values of α, The surface of $v(s,t)$ for $t \in (-0.5, 0.5), x \in (-0.5, 0.5)$

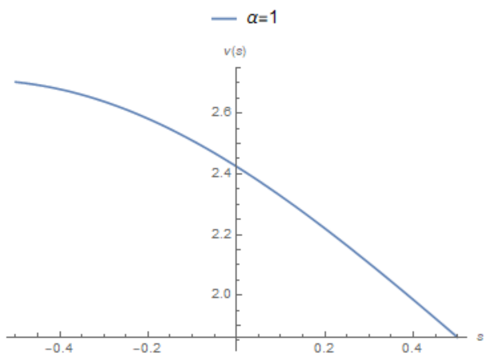


Fig:8a

Corresponding 2D figure for t=0.5 and α=

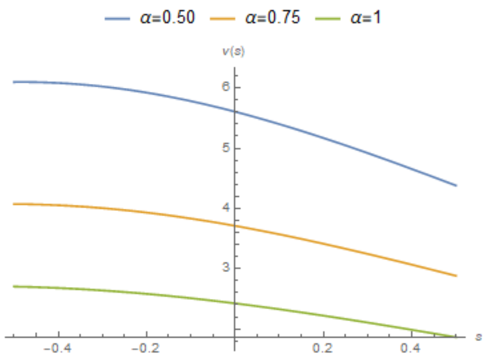


Fig:8b Corresponding fractional graph for different values of α, Corresponding 2D figure for t=0.5

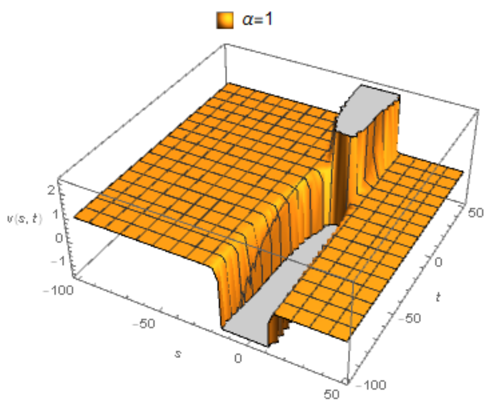


Fig:9a

The surface of v(s,t)for α=1, t∈(−100,50),x∈(−100,50)

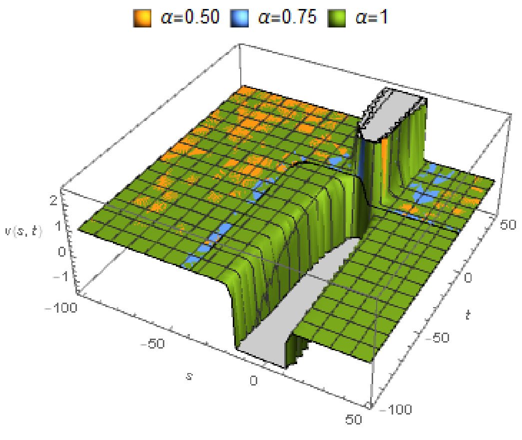


Fig:9b

Corresponding fractional graph for different values of α and t∈(−100,50),x∈(−100,50)

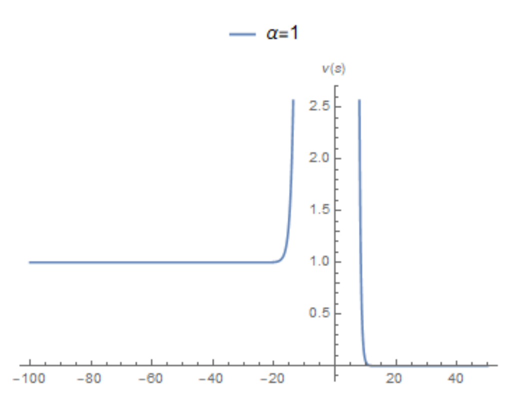


Fig:10a

Corresponding 2D figure for t=50 and α=1

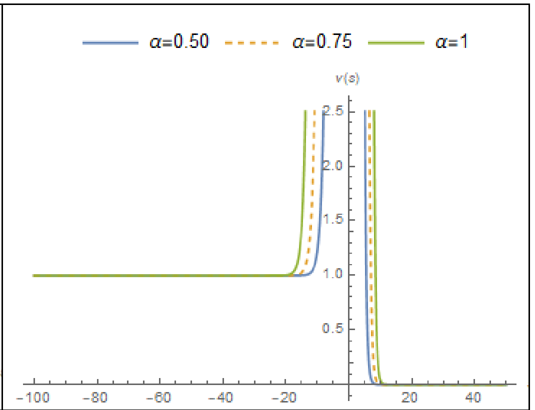


Fig:10b

Corresponding 2D figure for different values of α and t=50

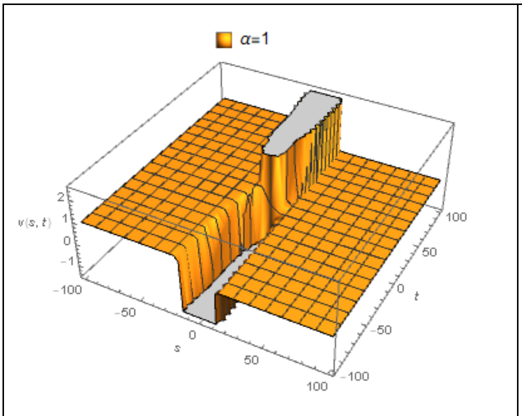


Fig:11a

The surface of v(s,t)for α=1, $t\in(-100,100), x\in(-100,100)$

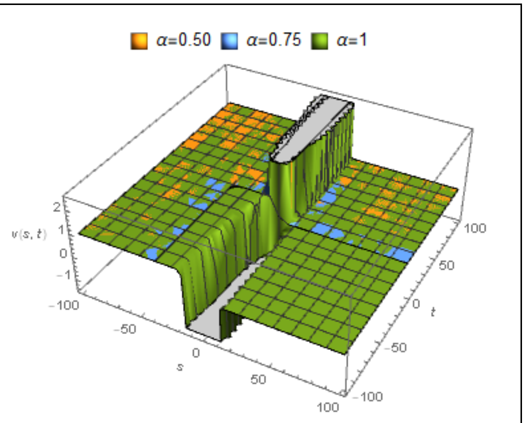


Fig:11b

Corresponding fractional graph for different values of α and $t\in(-100,100), x\in(-100,100)$

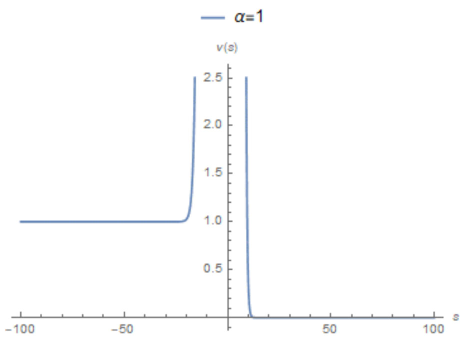


Fig:12a

Corresponding 2D figure for t=100 and α=1

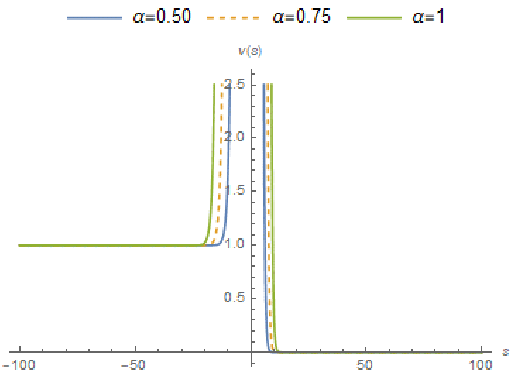


Fig:12b

Corresponding 2D figure for different values of α and t=100

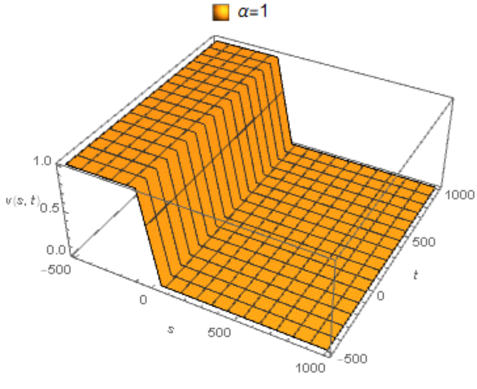


Fig: 13a. The surface of v(s,t)for α=1, t∈(−500,1000),x∈(−500,1000)

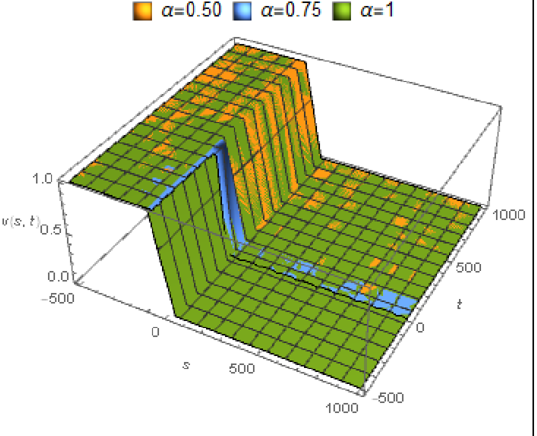


Fig:13b

Corresponding fractional graph

for different values of α and t∈(−500,1000),x∈(−500,1000)

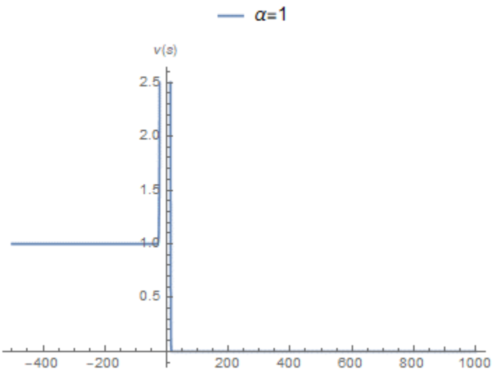


Fig:14a

Corresponding 2D figure for t=1000 and α=1

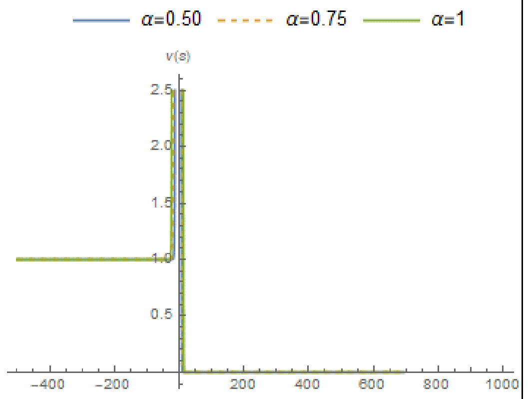


Fig:14b

Corresponding 2D figure for different values of α and t=1000

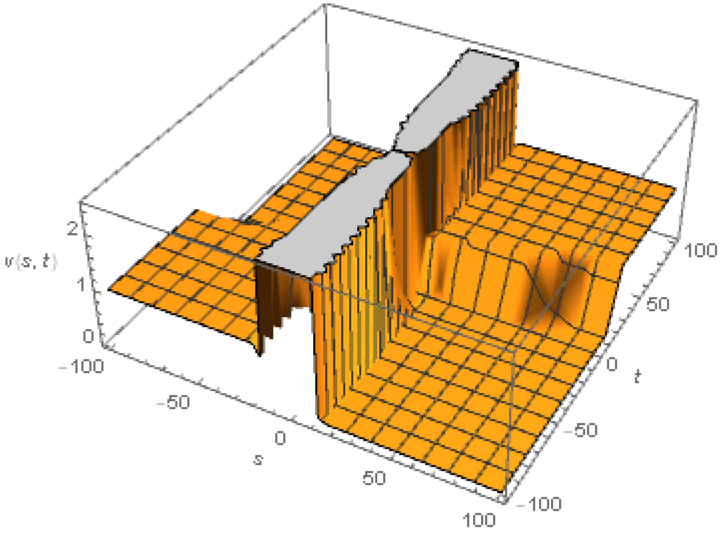


Fig:15b

The absolute error $t \in (-100,100), x \in (-100,100)$

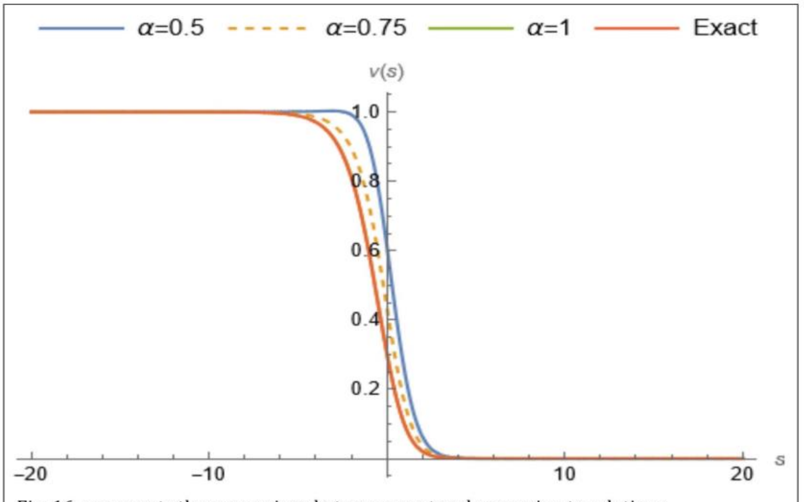


Fig: 16, represents the comparison between exact and approximate solutions.

4.4 Discussion for application # 2.

For a number of ranges of the variables s and t, the approximate findings are displayed graphically. Figures 7(a), 8(a), 9(a), 10(a), 11(a), 12(a), 13(a), 14(a), and 15(a) show the results in 2D and 3D For $\alpha = 1$. Figures 7(b), 8(b), 9(b), 10(b), 11(b), 12(b), 13(b), 14(b), and 15(b) show the results in fractional analysis in 2D and 3D for various values of α. Figure 15b represents the three-dimensional behavior and displays the absolute error.

Table#1, shows the comparison between homotopy perturbation method and the exact solution. The errors are very small in this table. It is clear from figure 16, that the approximate solution and the exact solution are very close for the chosen values of t. The results provide very strong evidence that is the homotopy perturbation technique is easy to get approximate solution of nonlinear equation. It is to be noted that four terms only were used in evaluating the approximate solutions.) $t \in (-100,100)$ and $s \in (-100,100)$.

Table # 1, of application 2, which represents the solution for different values of α with exact and error solution.

X	$\boldsymbol{\alpha = 0.50}$	$\boldsymbol{\alpha = 0.75}$	$\boldsymbol{\alpha = 1}$	EXACT	ERROR
0	0.711696	0.39917	0.304473	0.3023175	0.0021559087327
1	0.302996	0.141534	0.096436	0.096122	0.000319644942
2	0.078208	0.0319714	0.020134	0.020122	0.000011917301

3	0.014288	0.005436	0.003285	0.003286	0.0000012095912
4	0.002186	0.000806	0.000478	0.000479	$3.0951148303 \times 10^{-7}$
5	0.0003099	0.000113	0.0000666	0.000067	$4.8753865429 \times 10^{-8}$
6	0.0000427	0.0000155	0.0000091	0.0000091	$6.9273394389 \times 10^{-9}$
7	0.0000059	0.0000025	0.0000012	0.0000012	$9.5362944770 \times 10^{-10}$
8	7.88×10^{-7}	2.853×10^{-7}	1.676×10^{-7}	$\boldsymbol{\alpha = 0.50}$	$1.2985671901 \times 10^{-10}$
9	1.06×10^{-7}	3.864×10^{-8}	2.269×10^{-8}	2.271×10^{-8}	$1.7613742871 \times 10^{-11}$
10	1.44×10^{-8}	5.230×10^{-9}	3.072×10^{-9}	3.074×10^{-9}	$2.3857417347 \times 10^{-12}$

5 Conclusion

This study demonstrates the successful development of Homotopy Perturbation Method (HPM) for addressing fractional order nonlinear partial differential equations with different initial conditions. A notable strength of HPM lies in its capability to address inequalities without the need for discretionary interventions, perturbation transformations, or linearization's. The accuracy of HPM is evaluated through the presentation of results for various sample numbers, affirming its effectiveness in solving nonlinear fractional partial differential equations. The findings underscore the precision and efficiency of the method, establishing it as a reliable tool in the realm of solving complex nonlinear equations with fractional components.

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